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On the non-separable baroclinic parallel flow instability problem

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Perturbation series are developed and mathematically justified, using a straightforward perturbation formalism (that is more widely applicable than those given in standard textbooks), for the case of the two-dimensional inviscid Orr–Sommerfeld-like eigenvalue problem describing quasi-geostrophic wave instabilities of parallel flows in rotating stratified fluids.

The results are first used to examine the instability properties of the perturbed Eady problem, in which the zonal velocity profile has the form $u = z + \mu u_1(y, z)$ where, formally, $\mu \ll 1$. The connexion between baroclinic instability theories with and without short wave cutoffs is clarified. In particular, it is established rigorously that there is instability at short wavelengths in all cases for which such instability would be expected from the 'critical layer' argument of Bretherton. (Therefore the apparently conflicting results obtained earlier by Pedlosky are in error.)

For the class of profiles of form $u = z + \mu u_1(y)$, it is then shown from an examination of the $O(\mu)$ eigenfunction correction why, under certain conditions, growing baroclinic waves will always produce a counter-gradient horizontal eddy flux of zonal momentum tending to reinforce the horizontal shear of such profiles. Finally, by computing a sufficient number of the higher corrections, this first-order result is shown to remain true, and its relationship to the actual rate of change of the mean flow is also displayed, for a particular jet-like form of profile with *finite* horizontal shear. The latter detailed results may help to explain at least one interesting feature of the mean flow found in a recent numerical solution for the wave régime in a heated rotating annulus.

1. Introduction

This paper considers some fundamental aspects of the quasi-geostrophic baroclinic instability problem. Apart from its frequent relevance in laboratory situations involving slow motions of an inhomogeneous fluid in a rotating frame of reference, this parallel-flow instability problem yields a theoretical description of processes known to be important in the earth's atmosphere. It has already been studied extensively (see Pedlosky 1964*a*, Fowlis & Hide 1965). Simple baroclinic instability theory accounts qualitatively for the way in which many large-scale weather systems obtain their kinetic energy. For readers not familiar with the type of dynamics involved, a brief description is given in appendix A.

Of particular interest are forms of the relevant two-dimensional eigenvalue problem, (2.1) below, that are *non-separable* because of the presence of horizontal as well as vertical shear in the mean zonal current $\{u(y,z),0,0\}$. It is precisely such problems that can be expected to model the interesting kind of simultaneous potential and kinetic energy transformation known to play a role in the maintenance of the mid-latitude westerlies, the associated horizontal eddy flux of zonal momentum corresponding to a negative Austausch coefficient.

But the mathematical difficulties have for some time remained a serious theoretical obstacle. Valuable progress has been made with 'two-level models', equivalent to use of the crudest possible finite differencing in the vertical (Eliasen 1961; Pedlosky 1964*b*); the important recent work of Stone (1969), using such a model, will be mentioned later. Another approach that has been used, e.g. by Eady (see Green 1970) and by Brown (1969*a*), is to solve for particular numerical cases by the use of purely finite-difference methods of relatively high resolution. These avoid *a priori* assumptions about the vertical structure, but do not easily yield generality or insight.

In this paper we present a perturbation formalism (§ 3) that was developed in order to provide a flexible analytical approach to the non-separable eigenvalue problem (2.1). The method is applied to a discussion of the perturbed Eady problem, in which $u(y,z) = z+\mu u_1(y,z)$, μ being the perturbation parameter.

The idea of perturbing about a simple form of (2.1) is not new, being implicit for instance in some unpublished work of Stern & Magaard (see Magaard 1963), and having also been put forward by Pedlosky (1965). In the latter's investigation, first correction terms for the perturbed Eady problem were obtained for small μ by means of a somewhat elaborate initial-value approach, which brings in the (singular) complete set of unperturbed eigenfunctions in a way reminiscent of classical perturbation theory. By contrast, the present approach is relatively straightforward, and we can obtain the higher corrections as well, yielding results valid for a *finite* range of μ. An example for which such calculations were carried out in detail is given in § 7 below. A more fundamental consequence is that knowledge of the higher corrections enables us to justify our procedure in a mathematically rigorous way (§ 4).

The latter point gains added importance in view of the fact that Pedlosky's (1965) conclusions on the perturbed stability properties turn out to be in error (although that, indeed, becomes clear upon comparison with the numerical results found by Green 1960, for a particular example).

In § 5 we re-examine the perturbed stability properties. The main results have already been predicted by Bretherton (1966*a*, p. 333), using an indirect but powerful argument in which the idea of the 'critical layer' plays a key role. Our analysis can be looked upon as providing a mathematically rigorous expression of, and so a clear justification of, Bretherton's argument.

In § 6 we go on to develop a linear-theory explanation of the previously mentioned negative Austausch coefficient, for profiles of the form $u = z+\mu u_1(y)$ where $u_1(y)$ is an unspecified function. The discussion is based on the $O(\mu)$ terms, and identification of corresponding physical effects. Evidence that higher terms do not change the qualitative picture in cases of interest is then obtained (§ 7) by computing a sufficient number of terms in two 'realistic' examples, for which

$$u = z + \begin{Bmatrix} 0\cdot 4 \\ 0\cdot 5 \end{Bmatrix} \sin^2 \pi y \quad (0 \leqslant (y, z) \leqslant 1).$$

It is worth mentioning that the results for the first example have recently been compared, and show excellent detailed agreement, with corresponding results independently obtained by Brown (personal communication), using his finite-difference procedure (Brown 1969*a*).

2. The eigenvalue problem

Attention will be focused on a problem that is idealized but adequately embodies the fundamental properties under discussion. As will become clear, refinement would be essentially straightforward.

Consider small-amplitude frictionless adiabatic disturbances to a parallel flow $u(y, z)$ of stably-stratified Boussinesq liquid, whose horizontally averaged buoyancy or Brunt–Väisälä frequency is $N(z)$. The flow is in the x-direction and is limited by boundaries at $z = 0, H$ and $y = 0, L$, on which the normal velocity must vanish; x, y, z are Cartesian co-ordinates in a frame of reference rotating about the vertical z-axis with angular velocity $\frac{1}{2}f$. A dimensionless combination of importance in the problem is

$$\epsilon(z) \equiv \frac{f^2 L^2}{N^2 H^2},$$

which is formally of order unity, expressing the anticipated importance of both buoyancy and Coriolis forces. (But with our definition of L, numerical values of ϵ are more like π^2 in cases of interest.) Compressibility would introduce no essential modification provided that H is very much less than the density scale height, and the presence of a horizontal rotation component will have negligible effect if $H/L \ll 1$. Also, very crudely, one could regard the rigid upper boundary as the beginning of e.g. an idealized 'stratosphere of infinite static stability'.

The well known eigenvalue problem for the perturbation pressure

$$\mathrm{Re}\, \varphi(y, z)\, e^{ik(x-ct)}$$

of a quasi-geostrophic normal-mode wave disturbance can be written in dimensionless form, as

$$(u-c)\,[(\epsilon\varphi_z)_z + \varphi_{yy} - k^2\varphi] + q_y(y, z)\,\varphi = 0, \tag{2.1a}$$

$$(u-c)\,\varphi_z - u_z \varphi = 0 \quad \text{on} \quad z = 0, 1, \tag{2.1b}$$

$$\varphi = 0 \quad \text{on} \quad y = 0, 1, \tag{2.1c}$$

where the dimensionless wave-number k is real and the complex amplitude $\varphi(y, z)$ and phase velocity c are sought as eigenfunction and eigenvalue. The function $q_y(y, z)$, a property of the basic flow analogous to the $-u_{yy}$ of the classical Orr–Sommerfeld problem, is defined in appendix A (ii), which briefly sketches the derivation of (2.1). The boundary conditions (2.1*b*) reflect the active role that can be played by horizontal boundaries in the present problem, due to the importance of vertical vortex-tube stretching.

Since $(u-c)^{-1}q_y$ (see (A 6)) is not in general of the form func (y) + func (z), nor $(u-c)^{-1}u_z$ independent of y, the problem is generally non-separable in the coordinates y, z. Certain neutral separable solutions (c real) are possible if u is of the form func $(y)\times$ func (z), and β (see (A 6)) is zero, but are not of great importance in themselves.

As is known from particular solutions, such a system can exhibit more than one type of instability (Phillips 1963, § 3a), but in speaking of the 'baroclinic instability problem' one is thinking of situations in which the vertical shear u_z is the most essential feature of the basic velocity profile. Because of hydrostatic and geostrophic (pressure-Coriolis) balance, the vertical shear is associated with a transverse horizontal density gradient, and thus represents a store of available potential energy (Lorenz 1955). Under suitable conditions some of this mean-flow energy can be released by a growing disturbance in the manner described in appendix A (i), as was first clearly shown by the independent mathematical analyses of Charney (1947), and Eady (1949) (see appendix A (iii)).

We shall take (2.1) as our starting point. Although, in view of the aforementioned controversy, some care will be taken to construct solutions of (2.1) in a mathematically rigorous way, one should remain aware that (2.1) is already the result of several formal approximations. (But it can be noted that our perturbation approach could be used as the basis for a mathematical justification of the latter too, if desired; in that connexion see the footnote to appendix D (ii).)

The assumption of normal-mode form for the solutions needs little discussion here, because the results we shall be interested in concern positive cases of instability. That the existence of instability in the normal-mode sense must imply instability in the solution to the general initial-value problem hardly needs proof, but in any case the kind of analysis needed is not essentially different from that given e.g. by Pedlosky (1964c) and Burger (1966).

3. The perturbation formalism

The convergent perturbation series to be obtained below are based simply upon the use of a generalized Green's function (Courant & Hilbert 1953, p. 354). This device seems more natural, and is certainly much more widely applicable, than the standard eigenfunction expansions elaborated upon in textbooks on theoretical physics. In particular, it does not depend upon a complete set of unperturbed eigenfunctions; cf. Courant & Hilbert (1953, p. 343), Morse & Feshbach (1953, p. 1034), Pedlosky (1965, § 1).

The problem to be considered in detail in this paper is that for which

$$u = z+\mu u_1(y,z), \quad q_y = \mu q_{1y}(y,z), \tag{3.1}$$

where μ is the perturbation parameter. That is, we shall be perturbing about an Eady solution, (A 7).

Note from (A 6) that a transverse gradient of the Coriolis parameter, or β-effect, can be included, as long as β can be written in the form $\mu\beta_1$ within the radius of convergence of the perturbation scheme. For simplicity, $\epsilon(z)$ will be assumed to remain constant. But it should be realized that there would be no formal difficulty in writing $\epsilon = \text{const.} + \mu\epsilon_1(z)$, or in perturbing about any other (e.g. a separable neutral) solution, etc., etc.

Because of the branch points in the k-dependence of the Eady solutions at $k = k_N$ (A 8), it turns out that two cases must be considered separately, namely $k \neq k_N$ and $k = k_N$.

(i) *The case* $k \neq k_N$

It seems natural to pose

$$\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \dots. \tag{3.2a}$$

Regarding c as the eigenvalue and the other parameters as fixed, one would also expect that

$$c = c_0 + \mu c_1 + \mu^2 c_2 + \dots. \tag{3.2b}$$

On substituting (3.1) and (3.2) into (2.1) and equating like powers of μ, a succession of boundary value problems is obtained, whose details are given in appendix B. Here we abbreviate the lth problem to

$$\begin{cases} L(\varphi_l) = I_l, & (3.3a)\\ D(\varphi_l) = B_l' + \dfrac{c_l\varphi_0}{(z-c_0)^2}, \equiv B_l, \quad \text{on} \quad z = 0, 1, & (3.3b)\\ \varphi_l = 0 \quad \text{on} \quad y = 0, 1, & (3.3c) \end{cases}$$

where

$$L \equiv \frac{\partial^2}{\partial z^2} + \frac{1}{\epsilon}\left(\frac{\partial^2}{\partial y^2} - k^2\right),$$

$$D \equiv \frac{\partial}{\partial z} - \frac{1}{z-c_0}.$$

Of course $I_0 = B_0 = 0$, so that $\{\varphi_0, c_0\}$ is an Eady mode. For $l \geqslant 1$, I_l and B_l' involve $\varphi_0, \dots, \varphi_{l-1}$ and $c_0, \dots, c_{l-1}$ only, as can be verified from (B 2).

Now the homogeneous problem complementary to (3.3), for $l \geqslant 1$, is the same as the zero-order problem (and its adjoint), and has the non-trivial solution φ_0. This means that the inhomogeneous problem (3.3) has a solution only if the inhomogeneity $\{I_l, B_l\}$ satisfies a certain condition of orthogonality to φ_0 (more generally, to the corresponding solution of the adjoint problem). It is that condition, of course, that determines c_l at each stage.

What the orthogonality condition must be can be found by formally multiplying (3.3a) by φ_0, integrating from 0 to 1 with respect to y and z, integrating by parts twice, and then using the boundary conditions and the fact that $L(\varphi_0) = 0$. Thence

$$-\iint \varphi_0 I_l \,dy\,dz + \int dy \left[\varphi_0 B_l\right]_{z=0}^{z=1} = 0.$$

Referring to (3.3b) and using the identity (A 13) we can rewrite this as an explicit formula for c_l:

$$c_l = \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})}\left\{\iint \varphi_0 I_l \,dy\,dz - \int dy \left[\varphi_0 B_l'\right]_{z=0}^{z=1}\right\}. \tag{3.4a}$$

Here φ_0 is normalized as in (A 7). For $l = 1$ (3.4a) gives the first correction for c, in terms of φ_0 only; it will be discussed in detail in § 5.

Equation (3.4a) was derived as a necessary condition for (3.3) to have a solution. It is also sufficient; when (3.4a) is satisfied a solution is

$$\varphi_l = -\iint \mathfrak{G}(y,z;\,\eta,\zeta)\, I_l(\eta,\zeta)\, d\eta\, d\zeta + \int d\eta \Big[\mathfrak{G}\,.\, B_l(\eta,\zeta)\Big]_{\zeta=0}^{\zeta=1}, \tag{3.4b}$$

where $\mathfrak{G}(y,z;\,\eta,\zeta)$ is a Green's function in the generalized sense (Courant & Hilbert 1953, p. 354). Here $\mathfrak{G}$ is a solution of

$$L(\mathfrak{G}) = A\varphi_0(y,z)\,\varphi_0(\eta,\zeta) - \delta(y-\eta)\,\delta(z-\zeta), \tag{3.5a}$$

$$D(\mathfrak{G}) = 0 \quad \text{on} \quad z = 0, 1, \tag{3.5b}$$

$$\mathfrak{G} = 0 \quad \text{on} \quad y = 0, 1. \tag{3.5c}$$

L and D are understood to operate on (y,z) and A is a constant defined so that (3.5) is soluble. It can be seen that A is given by

$$A\iint \varphi_0^2\, dy\, dz = 1.$$

As yet there is arbitrariness in $\mathfrak{G}$ and φ_l to the extent that a constant multiple of φ_0 may be added. (This corresponds to multiplying $\varphi = \Sigma\mu^l\varphi_l$ by a constant, $1+O(\mu)$.) It seems natural in the present context to remove this arbitrariness by requiring that φ_l be 'as small as possible', in terms of a norm such as

$$\left\{\iint \varphi_l\varphi_l^*\, dy\, dz\right\}^{\frac{1}{2}}.$$

The asterisk denotes the complex conjugate. That norm is minimized when φ_l satisfies

$$\iint \varphi_l\varphi_0^* = 0.$$

Correspondingly, we shall choose $\mathfrak{G}$ so that, for all η, ζ,

$$\iint \mathfrak{G}(y,z;\,\eta,\zeta)\,\varphi_0^*(y,z)\, dy\, dz = 0. \tag{3.6}$$

$\mathfrak{G}$ is uniquely defined by (3.5) and (3.6); an explicit representation is given in appendix B.

In summary, the solution to the perturbed problem is given formally by

$$\{\varphi, c\} = \left\{\sum_0^\infty \mu^l\varphi_l,\quad \sum_0^\infty \mu^l c_l\right\},$$

where all the terms for $l \geqslant 1$ are defined by (3.4), together with the recursion formulae written out in appendix B.

(ii) *The case $k = k_N$.*

Equation (3.4a) shows that the series just derived fail if $c_0 = \frac{1}{2}$, i.e. at the critical neutral wave-number $k = k_N$. But expansions in powers of $\lambda \equiv \mu^{\frac{1}{2}}$ turn out to be appropriate:

$$\varphi = \varphi_0^N + \lambda\varphi_1^N + \lambda^2\varphi_2^N + \ldots, \tag{3.7a}$$

$$c = \tfrac{1}{2} + \lambda c_1^N + \lambda^2 c_2^N + \ldots, \tag{3.7b}$$

where φ_l^N and c_l^N are not to be confused with the previous lth coefficients in the μ expansions. Although there are some features of interest, the essential ideas are the same, and the details are relegated to appendix C.

Dependence of c_l on the lower eigenfunction corrections

As is easy to verify, (3.4a) depends on $\varphi_0, \varphi_1, \ldots, \varphi_{l-1}$. Although (3.4$a$) is the simplest and most convenient form for computational purposes, it can be noted that as a consequence of the self-adjointness of L, c may be found in terms of $\varphi_0, \ldots, \varphi_{[\frac{1}{2}l]}$ only, where $[\frac{1}{2}l]$ denotes the largest integer $\leqslant \frac{1}{2}l$. That can be shown by first forming the equation

$$\iint \sum_{j=0}^{[\frac{1}{2}l-\frac{1}{2}]} \{\varphi_j(3.3\,a)_{l-j} - \varphi_{l-j}(3.3\,a)_j\}\,dy\,dz,$$

and then integrating by parts, using the boundary conditions. Alternatively and more elegantly (L. Segel, private communication; Morse & Feshbach 1953), the result can be derived from variational considerations. This generalizes a result given by Joseph (1967).

4. Mathematical interpretation and justification of the formulae

I_l is singular at $z = c_0$, and so it is necessary to define the meaning of expressions such as (3.4a,b) when the unperturbed eigenvalue c_0 is real, as is the case for $k \geqslant k_N$.

That is easy if we assume that u_1 and q_{1y} are analytic functions. The whole process can then be carried out in a domain $\mathscr{D} \equiv \Gamma_z \times \Gamma_y$, where Γ_y is the interval $0 \leqslant y \leqslant 1$, and Γ_z is a contour in the complex z-plane which joins 0 and 1 and avoids $z = c_0$ (see figure 1). Γ_z could depend on y, and must also be chosen so that u_1 has no singularities between Γ_z and the real axis (the shaded region). Once an appropriate Γ_z has been chosen, then (3.4) (and (C 6)) are unambiguous even for real c_0. Clearly $\mathfrak{G}$ can be understood as being defined by (B 3) of appendix B; it is convenient to suppose that both z and ζ lie on Γ_z.

Having chosen an appropriate $\mathscr{D}$ (which need not be complex if c_0 is not real), one can prove, for $|\mu| <$ some finite positive μ_0, that the c expansion is convergent, that the φ expansion is uniformly convergent over $\mathscr{D}$, and that the expansions do in fact represent a solution in $\mathscr{D}$ of the original problem (2.1). The proof is quite straightforward but somewhat tedious. It is given in appendix D for $k \neq k_N$; the proof for $k = k_N$ is similar.

After the perturbed $c = \sum_0^\infty \mu^l c_l$ has been found for given μ within the radius of convergence, it must then be asked whether any point $z_c(y)$ for which $u(y, z) = c$ falls within the shaded region or on the real axis. If so, the eigensolution on $\mathscr{D}$ is generally not the continuation of a solution regular in the physical domain of real y and z. If not, it is. (That the eigenfunction φ, as opposed to the terms in its representation $\Sigma\mu^l\varphi_l$, cannot then be singular in, or on the boundary of, the shaded region, will probably be obvious to the reader. In any case, it could be proved using continuation, upon substituting for c (now known) in (2.1a) and e.g. for $(u-c)\,\varphi_z$ (known) in (2.1*b*), thus considering (2.1) as an inhomogeneous boundary value problem for φ in which $(u-c)^{-1}\,q_y$ can be taken as a known regular analytic function of z for each y.) In this latter case, φ being smoothly behaved for real y and z in $0 \leqslant (y,z) \leqslant 1$, it must represent a physically meaningful solution, and we shall speak of an 'admissible' eigensolution.†

If the perturbed c happened to fall exactly on the real axis, further discussion would be needed. However, that possibility seems to be of little interest.

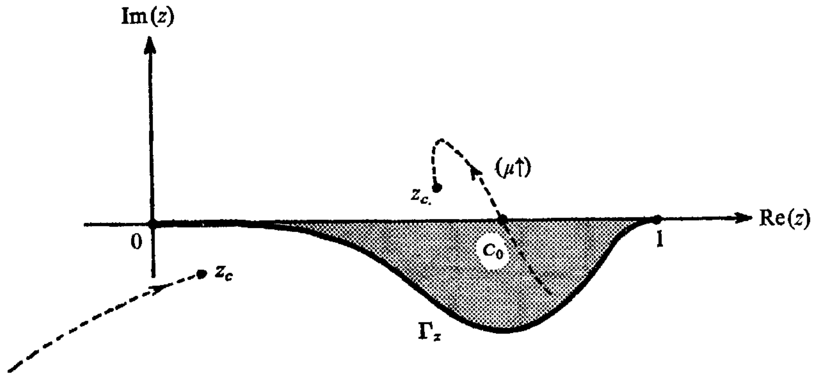


FIGURE 1. An example of an 'admissible' (for $\mu > 0$) configuration in the z plane; $\Gamma_z(y)$ must be chosen so that u_1, as a function of z, has no singularities in the shaded region for any y in Γ_y. Possible paths of the point z_c as μ varies are illustrated by the dotted lines. Note that $z_c(\mu)$ will in general be multivalued, and that that might possibly require extra care in the choice of Γ_z for a finite value of μ.

5. The first-order instability properties of the perturbed Eady problem

As a first, simple application, we discuss the stability of the mean flow $u = z+\mu u_1$, for $\mu \ll 1$ and any sufficiently differentiable function $u_1(y,z)$. It will appear below that the first-order results are qualitatively useful over a fair range of μ values of practical interest, especially for the short waves $k > k_N$, (A 8).‡ For $k \geqslant k_N$ it is also assumed that u_1 is analytic.

† When solutions are found that have Im $(c) \neq 0$ and are admissible by our definition, they must occur in complex conjugate pairs, Im $(c) \gtrless 0$, corresponding to two appropriate choices of Γ_z. One solution is 'damped' (but not dissipative!) and the other amplifying. The latter solution is the physically interesting one, but to be self-consistent one should also admit the former under the definition; we emphasize this point only because confusion about it sometimes seems to occur in the literature. *Any* non-singular normal mode of an approximate (e.g. non-dissipative) problem must of course be expected to represent a physically meaningful approximate solution, in the natural sense that over a finite time interval it approximates a solution (as opposed to a normal mode) of whatever are being regarded as the exact equations, e.g. equations with small diffusion coefficients. See the related discussion by Lin (1961).

‡ It is sometimes said that the basic quasi-geostrophic approximation (appendix A (ii)) becomes invalid at short wavelengths. However, it can be shown that that is not the case; the physical reason is that the height scale of the wave (see appendix A (i)), and hence the mean flow vertical velocity difference seen by the wave, diminishes as the wavelength for large enough k.

When $k \neq k_N$, (3.2*b*) is the appropriate expansion. Referring to (3.4*a*), we have

$$c = c_0 + \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})} K\mu + O(\mu^2), \tag{5.1}$$

where, as can be verified from (B 2),

$$K = \iint \frac{-\epsilon^{-1} q_{1y} \varphi_0^2}{z - c_0} dy\, dz - \int dy \left[\frac{-u_1 + (z-c_0) u_{1z}}{(z-c_0)^2} \varphi_0^2 \right]_{z=0}^{z=1}. \tag{5.2}$$

From (A 6), q_{1y} is related to u_1 by

$$q_{1y} = \beta_1 - \epsilon u_{1zz} - u_{1yy}, \tag{5.3}$$

with the obvious definition of the constant β_1. Only the zero-order eigenfunction φ_0, given by (A 7), is involved in this first correction to c_0.

When $k = k_N$, (3.7*b*) is appropriate. More explicitly, it may be shown (appendix C) that for $k = k_N$

$$c = \tfrac{1}{2} \pm \left\{ \frac{\alpha_N^2 - 1}{8\alpha_N^4} K_N \mu \right\}^{\frac{1}{2}} + O(\mu), \tag{5.4}$$

where K_N is defined by (5.2), with $\varphi_0 = \varphi_0^N$ and $c_0 = \frac{1}{2}$, as in (A 10).†

When c_0 is real these formulae must be interpreted in accordance with the discussion of § 4. The most interesting thing about them is that, in the short-wave neutral régime $k \geqslant k_N$ of the zero-order problem, Im (c) is non-zero in general, even though c_0 and φ_0 are real. For $k > k_N$ the imaginary contribution to c comes entirely from the half-residue at $z = c_0$ of the first integrand in (5.2). Taking the Γ_z shown in figure 1, we have

$$\mathrm{Im}\,(c) = -\pi \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})} \int_0^1 \mu \epsilon^{-1} \{q_{1y} \varphi_0^2\}_{z=c_0} dy + O(\mu^2), \quad \text{when} \quad k > k_N. \tag{5.5}$$

Note that, to first order, Im (c) depends on q_{1y} at $z = c_0$ only. *Whenever* (5.5) *is positive, we have an admissible amplifying mode* (as well as its 'damped' conjugate, by the conjugate choice of Γ_z). *If* (5.5) *is negative, there is no admissible normal mode to which* φ_0 *is a first approximation.*

The half residue can be given an illuminating physical interpretation as a critical-layer quasi-potential-vorticity flux, following the discussion given by Bretherton (1966*a*). His argument shows clearly why instability is to be expected whenever $\int \mu q_{1y} \varphi_0^2 dy$ has the appropriate sign at the unperturbed critical level $z = c_0$.

The factor multiplying the integral in (5.5) is positive for the lower wave $c_0 < \frac{1}{2}$, and negative for the upper wave $c_0 > \frac{1}{2}$, as can most easily be seen from (A 13). Therefore, the lower wave is destabilized by a positive weighted-average

† The formula (5.1) could have been obtained very simply, although not rigorously, by the Tollmien argument (Lin 1955, p. 122). One then has to *assume* the existence of neighbouring eigensolutions. (Conversely, the present type of analysis justifies the Tollmien argument). It is possible to derive (5.4) in a similarly simple way if one is prepared to assume also, writing $c - \frac{1}{2} \equiv \Delta(k, \mu)$, that $(\partial\Delta^2/\partial\mu)_k = -(\partial\Delta^2/\partial k)_\mu\, (\partial k/\partial\mu)_\Delta$ at the singular point $\{k = k_N, \mu = 0\}$. (The latter relation is thus true, but its truth does not seem obvious *a priori*.)

quasi-potential vorticity gradient $\int \mu\epsilon^{-1} q_{1y}|_{z=c_0} \sin^2 m\pi y\, dy$, such as might, especially for $m = 1$, be associated with a horizontally-jet-like profile. The upper wave is destabilized by a negative gradient.

With regard to the interpretation of (5.4), it is evident that in virtue of the two sign possibilities in (5.4) for *each* Γ_z, there is always, in general, exactly one conjugate pair of admissible solutions at $k = k_N$. An exception occurs when $K_N \mu$ is real positive (which cannot be so unless $\int q_{1y} \sin^2 m\pi y\, dy = 0$ at $z = \frac{1}{2}$).

These results and the related arguments of Bretherton (1966*a*) greatly clarify the connexion between baroclinic instability theories with and without 'short wave cutoffs'; see also Bretherton (1966*b*). (This connexion, or seeming lack of it, had puzzled many investigators in the past.) They also show that the conclusion stated by Pedlosky (1965, abstract), that 'only the vertically anti-symmetric and horizontally symmetric component of the velocity deviation affects the stability of the flow' (to $O(\mu)$), is incorrect. As can be seen from our discussion, the vertically (and horizontally) symmetric component of q_{1y}, and hence of u_1 in general, is also involved, at short wavelengths.†

Examples

It is of interest first of all to apply (5.1) and (5.4) to the simple case $u_1 = 0$, $\mu q_{1y} = \beta =$ const., for comparison with the results of Green (1960; see also Garcia & Norscini 1969). A comparison is presented in figure 2 for $\mu\epsilon^{-1} q_{1y}\, (= \epsilon^{-1}\beta) = 1$. The y-dependence has been suppressed by replacing $\sin^2 m\pi y$ by $\frac{1}{2}$, and setting $m = 0$ elsewhere, so as to correspond to Green's y-independent formulation. The sign of q_{1y} is positive throughout the flow. Accordingly, the lower wave is destablized and the upper wave disappears. The first-order formulae are quite accurate for this finite perturbation, except at the longer wavelengths and, for $\mathrm{Re}\,(c)$, at $k = k_N$. As can be shown from symmetry, the first-order change in growth rate is zero for $k < k_N$.

It is not surprising that perturbing about the Eady solution does not yield the long-wave phenomena discovered by Green, since the β-effect is dominant in these very long waves, and cannot be regarded as a perturbation. Indeed, it may be verified in general that (5.1) is not uniformly valid near $\kappa = 0$, the first correction behaving like κ^{-2}. The 'critical' κ (cf. Garcia & Norscini 1969) is,

† Nor, it should be added, does the statement appear to be true for the long-wave end of the spectrum considered in Pedlosky's (6.6). It certainly seems inappropriate, in principle, because of the fact that the perturbation method is not uniformly valid in the limit of small total wave-number κ, as will be remarked upon shortly. The corresponding results of Green (1960) and Garcia & Norscini (1969) seem, in a practical sense, a sufficient counter-example. (Pedlosky's analysis does include the β-effect, via a trivial modification, and hence should relate to Green's problem for small β.)

It should perhaps be pointed out that the argument given in § 5 of Pedlosky's paper appears to be in error (F. P. Bretherton, private communication). It does not seem to be a straightforward matter to produce a corrected version. For instance, suppose that (c_N, c_N^*) is the conjugate pair of admissible perturbed eigenvalues that is found in general at $k = k_N$, as was indicated above. Then (cf. (3.5), etc., of the paper in question) the quantities $-k^2 c_N c_N^*$, $-ik(c_N + c_N^*)$, are *not* regular analytic functions of μ at $\mu = 0$ and $k = k_N$, except in certain special cases. That can be seen immediately from (5.4); see also appendix C.

however, qualitatively indicated by the condition $\mathrm{Re}\,(c_0+\mu c_1) = 0$, even though the $O(\mu)$ change in $\mathrm{Im}\,(c)$ is zero. Note that if the y-dependence is reintroduced, κ cannot approach zero; the limit $\kappa \to 0$ implies infinite zonal *and* meridional length scales, and so is not of very great interest in practice. (See footnote § 6.)

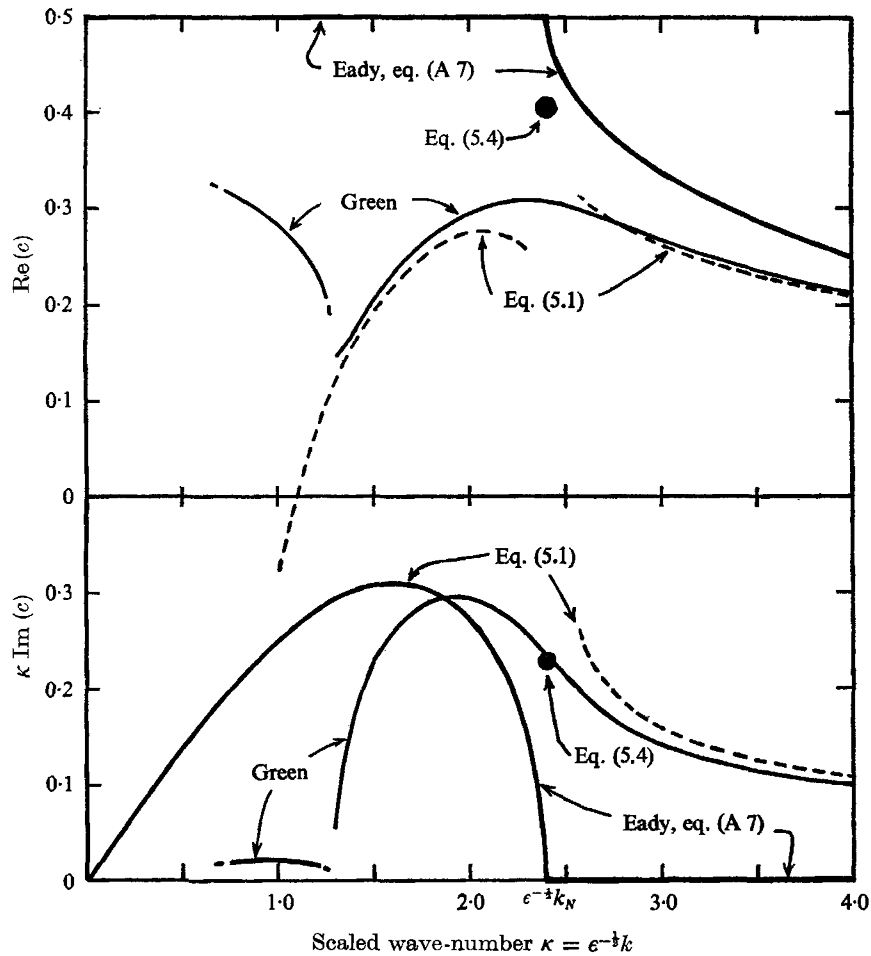


FIGURE 2. Comparison of first correction results from (5.1) and (5.4) with some numerical results of Green (1960), for $u = z$, $\epsilon^{-1}\beta = 1$, $m = 0$ (see text). Upper graphs: $\mathrm{Re}(c)$; lower: scaled growth rate $\kappa\,\mathrm{Im}\,(c) = \epsilon^{-\frac{1}{2}}k\,\mathrm{Im}\,(c)$. Note that the first correction to the growth rate is zero for $k < k_N$, but not for $k = k_N$(●) or $k > k_N$. For accuracy of comparison, the graphs of Green's results have been re-drawn, using his original data; in the case $\epsilon^{-1}\beta = \frac{1}{2}$ (not shown) the agreement at short wavelengths is even closer, upon correcting an inaccuracy in Green's corresponding published figure (*op. cit.*, p. 242; Green, private communication).

The formulae are illustrated further by the calculations presented in figure 3. There $\beta = 0$, $m = 1$, $\epsilon = 9$, and (*a*) $u = z_s(1-2y_s^2)$ and (*b*) $u = z_s - y_s^2$, where $y_s = (y-\frac{1}{2})$, $z_s = (z-\frac{1}{2})$.

In the first example, (a), it happens that $q_y = 0$ at $z = \frac{1}{2}$. There is a critical neutral mode (which, incidentally, is a separable solution). The corresponding wave-number k'_N in figure 3 was estimated from the formula

$$k'^2_N = k^2_N - \frac{4\epsilon K_N}{\alpha^2_N}\mu + O(\mu^2), \tag{5.6}$$

which can be established by the perturbation method (or again, obtained heuristically by the Tollmien argument). The perturbation formulae yield no admissible perturbed normal modes for $k > k_N$, at $O(\mu)$.

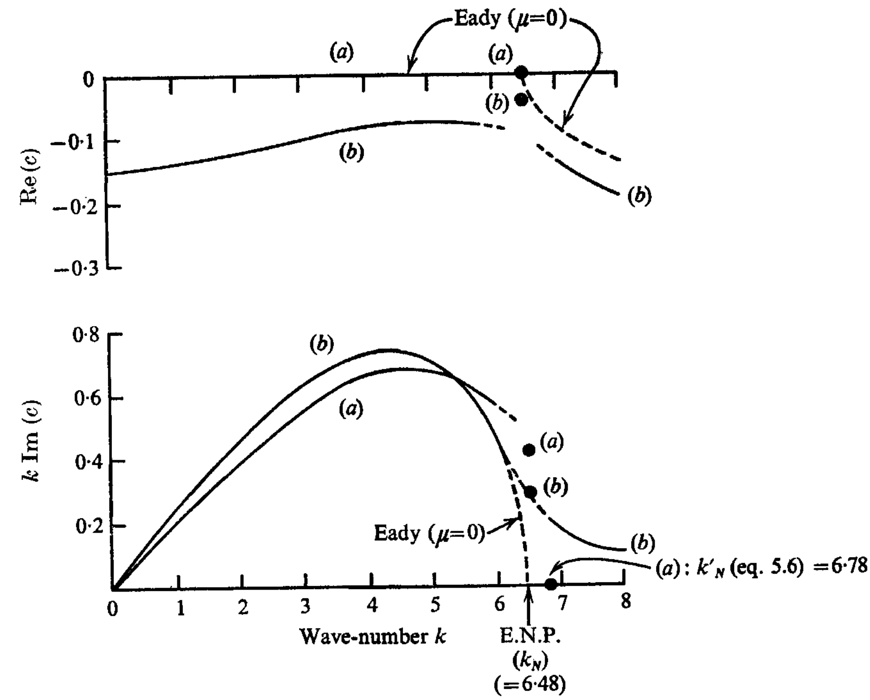


FIGURE 3. Examples of perturbed growth rate and phase velocity curves for the profiles

$$\left.\begin{array}{l}(a)\ u = z_s(1-2y_s^2)\\ (b)\ u = z_s - y_s^2\end{array}\right\} \quad -\tfrac{1}{2} \leqslant (y_s, z_s) \leqslant \tfrac{1}{2},$$

calculated from (5.1), except at the Eady neutral point (E.N.P.), where (5.4) and (5.6) were used (●). Note that $k = 0$ means that $\kappa = \epsilon^{-\frac{1}{2}}\pi$, not $\kappa = 0$. In these calculations $\epsilon/\pi^2 = 0{\cdot}912$, and $m = 1$ (gravest mode).

That probably means, in this example, that there are indeed no eigensolutions for $k > k'_N$, but that unstable modes exist for $k < k'_N$ even though the Eady neutral waves do not serve as first approximation to some of them. This tentative interpretation is confirmed by perturbing about the neutral separable solution (McIntyre 1967).

The second example (b) has a non-zero potential vorticity gradient at $z_s = 0$ as well as elsewhere. In this respect it is a less special case. The short wave instability appears in the same way, and for the same reason, as in Green's problem. Again, the correction to the growth rate is zero for $k < k_N$, by symmetry.

6. The tendency of baroclinic waves to generate a counter-gradient momentum flux

Consider a mean flow that is baroclinically unstable when $\mu = 0$ ($\epsilon > \pi^2/4\alpha_N^2$; see (A 8)) with horizontal shear that is independent of height:

Then (5.3) reduces to

$$\left.\begin{aligned} u(y,z) &= z+\mu u_1(y). \\ q_{1y} &= \beta_1 - u_{1yy}. \end{aligned}\right\} \qquad (6.1)$$

This includes the cases $u = z+(0{\cdot}4, 0{\cdot}5)\sin^2\pi y$ examined in detail in § 7, which bear sufficient qualitative resemblance (although that point should not be pushed too far) to zonal mean profiles both in the atmosphere (Lorenz 1967) and in laboratory analogues (Williams 1969; figure 7 *b, c* below) for one to hope for insights that are heuristically useful. For this mathematically simplest way of introducing horizontal shear it will prove easy to see, in quite an elegant degree of generality, its first-order effect on the horizontal wave structure and the associated momentum flux or Reynolds stress component, $-\overline{\rho u'v'}$.

The result that will be obtained below could be simply expressed by saying that, to first order, the horizontal phase of a gravest ($m = 1$) unstable Eady mode is distorted in the horizontal by the differential advection u_{1y} in the 'obvious' sense (see figure 4), and that that effect is guaranteed to predominate, in any given case, provided ϵ is greater than some number ϵ_0 ($> \pi^2/4\alpha_N^2$) formally of order unity. (This last form of proviso is always sufficient, but is not necessary in all cases, in particular when u_1 has the simple form ay^2+by implying that q_{1y} is constant. Since $\epsilon \propto L^2$ one may think of $\epsilon > \epsilon_0$ as meaning that the wave is not too closely constrained laterally.)

The result and its physical interpretation are not really obvious without the analysis, since the instability mode involves a subtle balance between advection and *propagation* effects. (Recall in that connexion that for a barotropic, or classical inviscid shear instability, the phase lines bend oppositely to the 'obvious way'.) The perturbation method permits an unambiguous discussion of how that balance is altered, under various circumstances, by introducing horizontal shear.

A horizontal structure of the kind illustrated in figure 4 is of interest because the associated Reynolds stress $-\overline{\rho u'v'}$ transports x-momentum *against* the mean gradient u_y, as can be seen immediately from figure 4*c*. Such a process is known to be important in the large-scale atmospheric zonal momentum balance in middle latitudes (Phillips 1963, p. 152).

The trend revealed by the $O(\mu)$ terms will be borne out by the finite μ calculations for $u_1 = \sin^2\pi y$ presented in § 7.

Before turning to details, we should point out that the stress $-\overline{\rho u'v'}$ is not the only significant mechanism of zonal momentum transfer, in the type of rotationally dominated flow under consideration. This point will be discussed in § 7. It is true, however, that for such flows the *vertically averaged* momentum transfer is described completely by the vertical average of $-\overline{\rho u'v'}$.

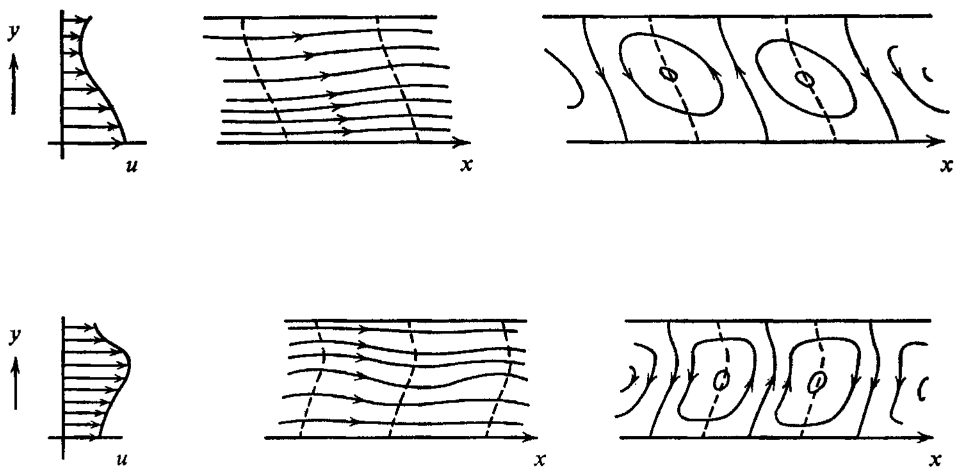
The expression (3.4b), with $l = 1$, may be written

$$\varphi_1 = \iint \mathfrak{G}\varphi_0(\eta,\zeta)\frac{\epsilon^{-1}q_{1\eta}(\eta,\zeta)}{\zeta - c_0}\,d\eta\,d\zeta$$

$$+\int d\eta\left[\mathfrak{G}\varphi_0(\eta,\zeta)\frac{-u_1+c_1+(\zeta-c_0)\,u_{1\zeta}}{(\zeta-c_0)^2}\right]_{\zeta=0}^{\zeta=1},$$

$$= \varphi_{1I}+\varphi_{1B}, \tag{6.2}$$

say. We are perturbing about an unstable wave, with $k < k_N$, and the expressions are uniformly valid in the real, physical domain.



(a) y-dependence of velocity profile (b) Total streamlines (c) Disturbance streamlines

FIGURE 4. Schematic diagrams representing the wave pattern in some given horizontal plane, for two hypothetical mean-flow velocity profiles $u = z + \mu u_1(y)$ whose y-dependences are depicted on the left.

In virtue of (A 2) and (A 5), the lines of constant phase in figure 4 are the same as the lines of constant phase of $\mathrm{Re}\,\{\varphi\, e^{ik(x-ct)}\}$ in a horizontal plane. The shape of the latter is given by the y dependence of $-\mathrm{ph}\,(\varphi)$, $= -\mathrm{ph}\,(\varphi_0+\mu\varphi_1)+O(\mu^2)$, at given z. It is convenient to consider the phase of $(\varphi_0+\mu\varphi_1)$ relative to the phase of φ_0, the latter phase being independent of y. Call this relative phase $\mu\Phi$; then

$$\mu\Phi(y,z) = \mathrm{ph}\,(\varphi_0+\mu\varphi_1)-\mathrm{ph}\,(\varphi_0) = \mu\frac{\mathrm{Im}\,(\varphi_0^*\varphi_1)}{|\varphi_0|^2}+O(\mu^2). \tag{6.3}$$

Consider the gravest mode $m = 1$. Take $u_1 = u_1(y)$, and substitute (6.2) into (6.3). The boundary contribution $-\Phi_B$ to $-\Phi$, arising from the second term in (6.2), may be written, using the form (B 3) for $\mathfrak{G}$ and recalling (A 7), as

$$-\Phi_B = -\frac{\mathrm{Im}\,(\varphi_0^*\varphi_{1B})}{|\varphi_0|^2}+O(\mu)$$

$$= \frac{1}{\sin \pi y}\sum_{n=2}^{\infty}\theta_n(z)\,C_n \sin n\pi y + \mathrm{func}\,(z)+O(\mu), \tag{6.4a}$$

where $$C_n = 2\int_0^1 [u_1(\eta)\sin\pi\eta]\sin n\pi\eta\, d\eta, \tag{6.4b}$$

and $$\theta_n(z) = \mathrm{Im}\left\{\frac{\chi_1^*(z)}{|\chi_1(z)|^2}\left[G_1^n(z;\zeta)\frac{\chi_1(\zeta)}{(\zeta-c_0)^2}\right]_{\zeta=0}^{\zeta=1}\right\}, \tag{6.4c}$$

the second (and irrelevant) contribution to (6.4a) being the $n = 1$ term.

Now if all the θ_n were positive and equal, (6.4a, b) would show at once that $-\Phi_B$, at each height z, would be exactly proportional to $u_1(y)$ (to within an additive function of z).

The θ_n are not equal, but it is possible to show after some manipulation of (6.4c), with the use of (B 3b) and (A 15), that for $k < k_N$ and given ϵ, z,

$$\theta_2 > \theta_3 > \theta_4 > \dots > 0. \tag{6.5}$$

In words, *the y-dependence of $-\Phi_B$ is qualitatively similar to, but more 'smoothed out' than, the y-dependence of u_1*, for any sufficiently simple $u_1(y)$. Note also that $\partial\Phi_B/\partial y = 0$ at $y = 0, 1$, as indeed must be true of $\partial\Phi/\partial y$ because of the boundary conditions. (In establishing (6.5) we use among other things the fact that $p^{-2} - p^{-1}\coth p < 0$, and $d(p^{-2} - p^{-1}\coth p)/dp > 0$ for $p > 0$.)

The remaining contribution to Φ, namely Φ_I, is given by another expression of the form (6.4a) with, say, C_n^I, θ_n^I instead of C_n, θ_n. The C_n^I are given by (6.4b) with $u_{1\eta\eta}$ instead of u_1, and

$$\theta_n^I(z) = \epsilon^{-1}\,\mathrm{Im}\left\{\frac{\chi_1^*(z)}{|\chi_1(z)|^2}\int_0^1 G_1^n(z;\zeta)\frac{\chi_1(\zeta)}{\zeta-c_0}\,d\zeta\right\}. \tag{6.6}$$

Let $u_1(y)$ be given, such that $\Sigma C_n \sin n\pi y\,(\neq 0)$ and $\Sigma C_n^I \sin n\pi y$ are uniformly and absolutely convergent (a very mild restriction), and either let κ_1 be held constant such that $\mathrm{Im}\,(c_0) \neq 0$ or, alternatively, fix attention on the fastest growing mode. Then we can prove that, uniformly in z, $(\max_y \Phi_I - \min_y \Phi_I)/(\max_y \Phi_B - \min_y \Phi_B) = O(\epsilon^{-1})$ as $\epsilon \to \infty$. This can be interpreted as implying that, *for $\epsilon >$ some ϵ_0 independent of z, the qualitative result obtained for Φ_B also applies to Φ*, as was to be shown. (Here one starts by establishing that, as $\epsilon \to \infty$ under the stipulated conditions, $\theta_n^I = O(1)$ uniformly in $n \geqslant 2$ and z, whereas, for any given $n \geqslant 2$, $\theta_n \geqslant \epsilon \times$ a positive quantity dependent on n but independent of ϵ, z.)†

Note that the u_1 contribution in (6.2) that gives rise to Φ_B does come from a term representing advection, by the mean flow, of the wave pattern; more precisely, of the disturbance 'boundary potential vorticity' (Bretherton 1966a, § 3).

We did not investigate whether or not Φ_I actually does tend to oppose Φ_B (as far as the y-dependence is concerned). A few numerical calculations suggest that, when $u_1 = \sin^2\pi y$, Φ_I does oppose Φ_B, at some but not necessarily all

† We remark that the singular limiting behaviour of Φ can be thought of as reflecting the physical unreality of coherence over width L of an Eady mode (A7) when $L \gg NH/f$, the dominant zonal wave-length. Even the slightest amount of horizontal differential advection $\mu u_1(y)$ will disorganize such a mode if the channel width is too large; conversely, if there *are* modes in the presence of slight horizontal shear, (A7) will no longer represent a first approximation to any of them (cf. Stone 1969). The same remark applies to the singular limit $\kappa \to 0$ mentioned in § 5.

heights z. It is noted here that the part of Φ_I due to an $O(\mu)$ β-effect is y-independent and therefore irrelevant. Also, if u_1 is of the special form ay^2+by, then all of Φ_I is y-independent and so irrelevant.

7. Some finite-μ results, and the effect on the mean flow

A pertinent example of the simple type of profile discussed in § 6 is

$$u = z+\mu \sin^2 \pi y. \tag{7.1}$$

We take $\beta = 0$, $\epsilon/\pi^2 = 1{\cdot}62$ ($\epsilon = 16$), and evaluate a number of terms of the series defined by (3.4), etc., for $m = 1$ and $k = k_M$, the zonal wave-number at which the zero-order solution (A 7) has maximum growth rate $k\,\mathrm{Im}\,(c_0)$ for the chosen value of ϵ. From appendix A (iii), $k_M = 6{\cdot}117$, $\kappa = 1{\cdot}719$, $c_0 = \frac{1}{2}+0{\cdot}179i$, $k_M\,\mathrm{Im}\,(c_0) = 1{\cdot}09$.

The first few terms of the c expansion are found to be

$$c = 0{\cdot}5+0{\cdot}179i+0{\cdot}591\mu-0{\cdot}110i\mu^2+0{\cdot}081\mu^3-0{\cdot}193i\mu^4+\dots. \tag{7.2}$$

Thus the growth rate $k_M\,\mathrm{Im}\,(c)$ is reduced by $O(\mu^2)$, and the phase velocity is increased. (Most of the $O(\mu)$ contribution to the latter, however, is merely a consequence of the increase in the average u as μ increases.) Presumably the wave we are considering is not the dominant wave for the modified profile. But the discussion in § 6 did not depend on the growth rate being maximized with respect to k, and there seems no general reason to believe that any essential features will be lost.

The c and φ expansions were summed for $\mu = 0{\cdot}4$ and $\mu = 0{\cdot}5$, giving, in particular, $c = 0{\cdot}746+0{\cdot}156i$ and $c = 0{\cdot}82_4+0{\cdot}14_0i$ respectively. For comparison, truncation to the terms exhibited in (7.2) gives $c = 0{\cdot}742+0{\cdot}156i$ and $c = 0{\cdot}806+0{\cdot}139i$ respectively, which already come close. Eleven terms were actually calculated, but the last few terms were not, and did not have to be, obtained very accurately. (As one might expect, the higher eigenfunction corrections take on an increasingly complicated spatial structure.) Their main use was as a check on convergence, which appeared safe to an accuracy of 1 or 2 % for the $\mu = 0{\cdot}4$ case,† although less good when $\mu = 0{\cdot}5$. Only the $\mu = 0{\cdot}4$ results will be presented in detail; those for $\mu = 0{\cdot}5$ are very similar.

† The independent finite-difference calculation by Brown mentioned in § 1, for $\mu = 0{\cdot}4$, was done on a 20×40 ($0 \leqslant y \leqslant \frac{1}{2}$, $0 \leqslant z \leqslant 1$) grid and agrees with our $\mu = 0{\cdot}4$ results to better than our roughly estimated accuracy. His c agrees with our value ($0{\cdot}746+0{\cdot}156i$) to three figures. The more stringent test of fitting $|\varphi|_{\max}$ and then comparing detailed results for $-(\overline{u'v'})_y$ gave $[-(\overline{u'v'})_y]_{\max} = 80{\cdot}3$ (cf. our value $81{\cdot}0$, 1 % higher) at bottom centre, and, at top centre, $-(\overline{u'v'})_y = 38{\cdot}9$ (cf. our value $38{\cdot}8$). The contour printout from which the contours in figure 5d below were drawn has a resolution of 41×21 points for the half space, and the contours it defines are consistent with Brown's grid point values at each point except for a negligible ($0{\cdot}2$ % of max.) inconsistency at the point $80y = 5$, $20z = 18$.

It should be pointed out that Brown's calculation shows one thing that ours cannot, namely that the wave under consideration is in fact the fastest growing quasi-geostrophic instability at the given value of k. (Brown's method amounts to integrating the linearized Fourier-transformed initial value problem.)

For $\mu = 0{\cdot}4$ the dimensionless growth rate is $k_M \operatorname{Im}(c) = 0{\cdot}95_5$, and figures 5*a* and 5*b* show the contours of modulus and negative relative phase of $\varphi(y, z)$. The dotted line is the mean-flow isotach for which $u = \operatorname{Re}(c)$, i.e. the critical 'level'.

Figure 5*b* is of particular interest. The surface represented by the contours can be thought of as a constant-phase surface in physical space, the x-axis being directed out of the paper, for disturbance pressure or streamfunction ψ' or, equally well, transverse velocity $v' = \psi_x$. The characteristic forwards–downwards slope is evident (cf. figure 8), indicating that the mode is still basically a baroclinic instability, as one would expect. But for a pure Eady wave the phase contours would exhibit no other feature, being horizontal straight lines.

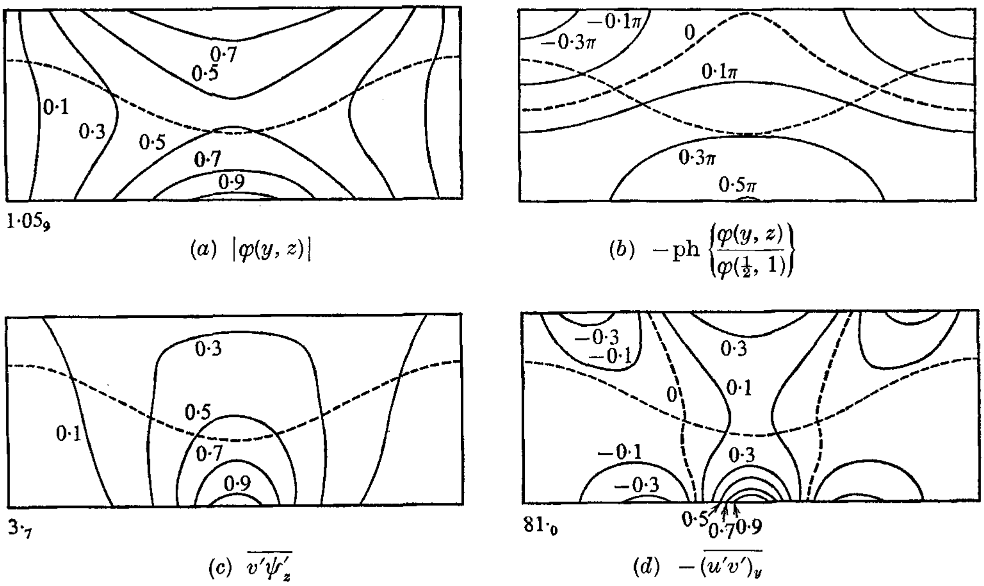


FIGURE 5. Distributions in the meridional or yz plane of quantities associated with an amplifying wave on the mean flow $u = z + 0{\cdot}4\sin^2\pi y$; $\beta = 0$, $\epsilon/\pi^2 = 1{\cdot}62$, $m = 1$, $k = k_M = 6{\cdot}12$. In (*a*), (*c*), and (*d*), the contour values are given as fractions of the maximum (dimensionless) value, shown on the left. The latter corresponds in each case to normalization as in (A 7) of the zeroth approximation φ_0. The dimensionalizing scales may be deduced from appendix A (ii). In (*d*), the eddy contribution to $\partial u/\partial t$, and (*c*), the transverse horizontal eddy flux of heat (i.e. buoyancy), the values are understood to be multiplied by the square of the actual (small) amplitude, times a factor $\exp\{2k \operatorname{Im}(c) t\}$. The thin dotted line in each diagram is the locus of points (y, z) such that $u(y, z) = \operatorname{Re}(c)$.

The actual structure in the horizontal is of the same general nature as that given by the first correction term (§ 6). The resulting Reynolds stress component is indicated by figure 5*d*, which plots the convergence $-\overline{(u'v')}_y$ or associated contribution to the zonal momentum tendency u_t. It is positive where the mean flow is already strongest.

Figure 5*c* shows the dimensionless transverse horizontal eddy heat flux $\overline{v'\psi'_z}$. It is in the positive y-direction, i.e. down the mean gradient, reflecting the fact that the growing wave, being a baroclinic instability, is drawing on mean flow potential energy.

The sharpness and prominence of the maximum at $z = 0$, $y = \frac{1}{2}$ in figure 5*d* is a higher order (finite μ) effect; in that respect the results go beyond what could have been expected from § 6. The recently published results of Brown (1969*a*), for a compressible atmosphere on a β-plane and a more complicated u profile in which the horizontal shear increases with height, show a behaviour that is similar in at least some respects. (Note the z-dependence of the y-average of Brown's $C(K_z, K_e) \equiv -u(\overline{u'v'})_y$ shown in his figure 5, in conjunction with u as given by his figure 1 and equation (3.1).)

The zonal momentum tendency

Although calculations of $-(\overline{u'v'})_y$ are suggestive by themselves, they do not actually give the second-order (in wave amplitude) rate of change of u, even though $-(\overline{u'w'})_z$ *is* negligible within the quasi-geostrophic approximation. To find u_t one must also take into account, explicitly or implicitly, the Coriolis force due to the slow mean meridional circulation that arises as a response to the strong dynamical requirement that mean-flow geostrophic and hydrostatic balance continue to hold (Eliassen 1952).

Calculations of u_t and of the stream function Ξ for the mean meridional circulation have been carried out. Their gross features are very much as would have been expected from the pioneering results of Phillips (1954) and Eliasen (1961) for the cruder two-level model. The mathematical framework involved is much the same as in those papers; details are given elsewhere (McIntyre 1967).

For comparison with the eddy contribution $-(\overline{u'v'})_y$ to it, u_t for the case $u = z + 0{\cdot}4\sin^2 \pi y$ is given in figure 6*a*; figure 6*b* shows the associated Ξ (see caption for details); note that the main part of the latter is thermally indirect.† Again, the results for $\mu = 0{\cdot}5$ are qualitatively similar.

The distribution of u_t (figure 6*a*) is recognizably similar to that of the eddy contribution (figure 5*d*). It is still positive around $y = \frac{1}{2}$, but considerably reduced at the top, $y = \frac{1}{2}$, $z = 1$, because of the Coriolis force associated with Ξ. The curiously sharp maximum at $y = \frac{1}{2}$, $z = 0$, is present also in the distribution of u_t.

This feature assumes considerable interest when we look at the mean zonal velocity field found in a recent numerical solution for the wave regime in a heated rotating annulus (Williams 1969). This is reproduced in figure 7*c* (see caption for details). If our result of figure 6 is a representative one, it shows in particular that baroclinic waves would initially tend to bring about a sharp peak in the horizontal

† This corresponds to a small positive rate of transformation of zonal mean kinetic into mean available potential energy, $C(K_z, A_z)$ (in the notation of Brown 1969*a*) $= +0{\cdot}4_3$ per unit zonal distance, in dimensionless units. Calling the disturbance energies K_e and A_e, the other customarily-defined energetic quantities have the values $C(A_z, A_e) = 14{\cdot}_4$, $C(A_e, K_e) = 6{\cdot}3_6$, $C(K_e, K_z) = 1{\cdot}0_3$; $\partial A_e/\partial t = 8{\cdot}_0$, $\partial K_e/\partial t = 5{\cdot}3_3$, $\partial A_z/\partial t = -14{\cdot}_0$, and $\partial K_z/\partial t = +0{\cdot}6_0$, so that the waves are bringing about a net *increase* in zonal mean kinetic energy, although at a rather small rate in this example. One might expect a greater rate at larger horizontal shear. Similar energy transformations are known to take place in the westerly wind systems of the earth's atmosphere.

dependence of u near the bottom centre of the cross-section. That is exactly what is seen in figure 7*c*; notice the kinks in the isotachs. It should also be mentioned that the finite amplitude waves found in the numerical solution are indeed very similar in spatial structure to the baroclinic waves of linearized theory (Williams, *op. cit.*).

It would be interesting to extend the calculations to incorporate the Ekman suction due to a lower frictional boundary layer. That would be straightforward (Barcilon 1964), but has not yet been done for this model (although the more sophisticated model studied by Brown 1969*b* incorporates such an effect, among many others). A significant gain in meteorological realism may however require other refinements as well, including non-linear effects (Smagorinsky 1964, p. 3; Thompson 1959, § 1).

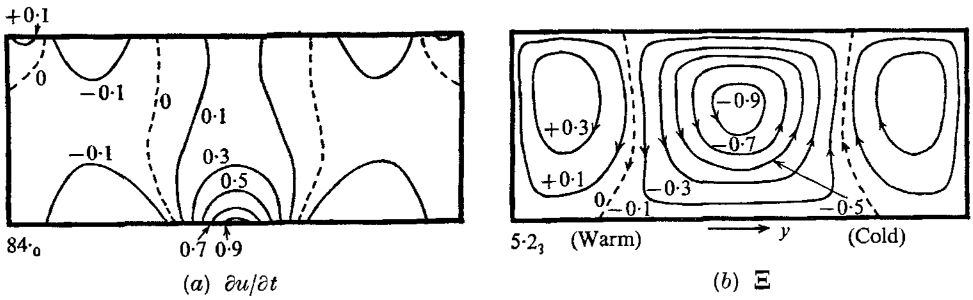


FIGURE 6. Effect of the amplifying wave on the mean flow, for the same case as in figure 5. (*a*) Dimensionless rate of change u_t of the mean zonal momentum. (*b*) Streamfunction Ξ of the associated mean meridional circulation ($w = \Xi_y$, $v = -\Xi_z$). The contour values are to be understood in the same sense as in figure 5. The dimensionalizing scale for Ξ is $Ro\, HU$, $\times$ (amplitude)$^2 \times \exp\{2k\, \mathrm{Im}\,(c)\, t\}$.

To relate the above results to the large scale motions of the atmosphere, note that if L is formally identified with 40° of latitude (although $\beta = 0$ in these calculations) and $H = 10\,\mathrm{km}$, and if the wave amplitude is such that the north–south velocity has amplitude $10\,\mathrm{m\,sec^{-1}}$ at the 'tropopause' and $12\,\mathrm{m\,sec^{-1}}$ at the ground, then u_t has respective values of 0·7 and $5{\cdot}6\,\mathrm{m\,sec^{-1}\,day^{-1}}$. This is of the right order of magnitude to be invoked as a partial explanation of the maintenance of the westerly winds (against the frictional retardation that is not included in our model). The zonal wavelength is about $4500 \times 2\pi/k_M \simeq 4500\,\mathrm{km}$, and if the vertical shear of the mean flow is $U/H = 2{\cdot}5\,\mathrm{m\,sec^{-1}\,km^{-1}}$, then the doubling time is $(\ln 2/k\,\mathrm{Im}(c))\,(L/U) = 1{\cdot}5\,\mathrm{day}$.

8. Concluding remarks

The perturbation method has proved to be a powerful tool, having permitted a significant degree of generalization of our precise mathematical knowledge of the non-separable problem (2.1), (appendix D), the computation of accurate and physically interesting solutions to it (§ 7) and, most important, physical insight into aspects of the processes it describes (§§ 5, 6), through interpretation of the first correction terms.

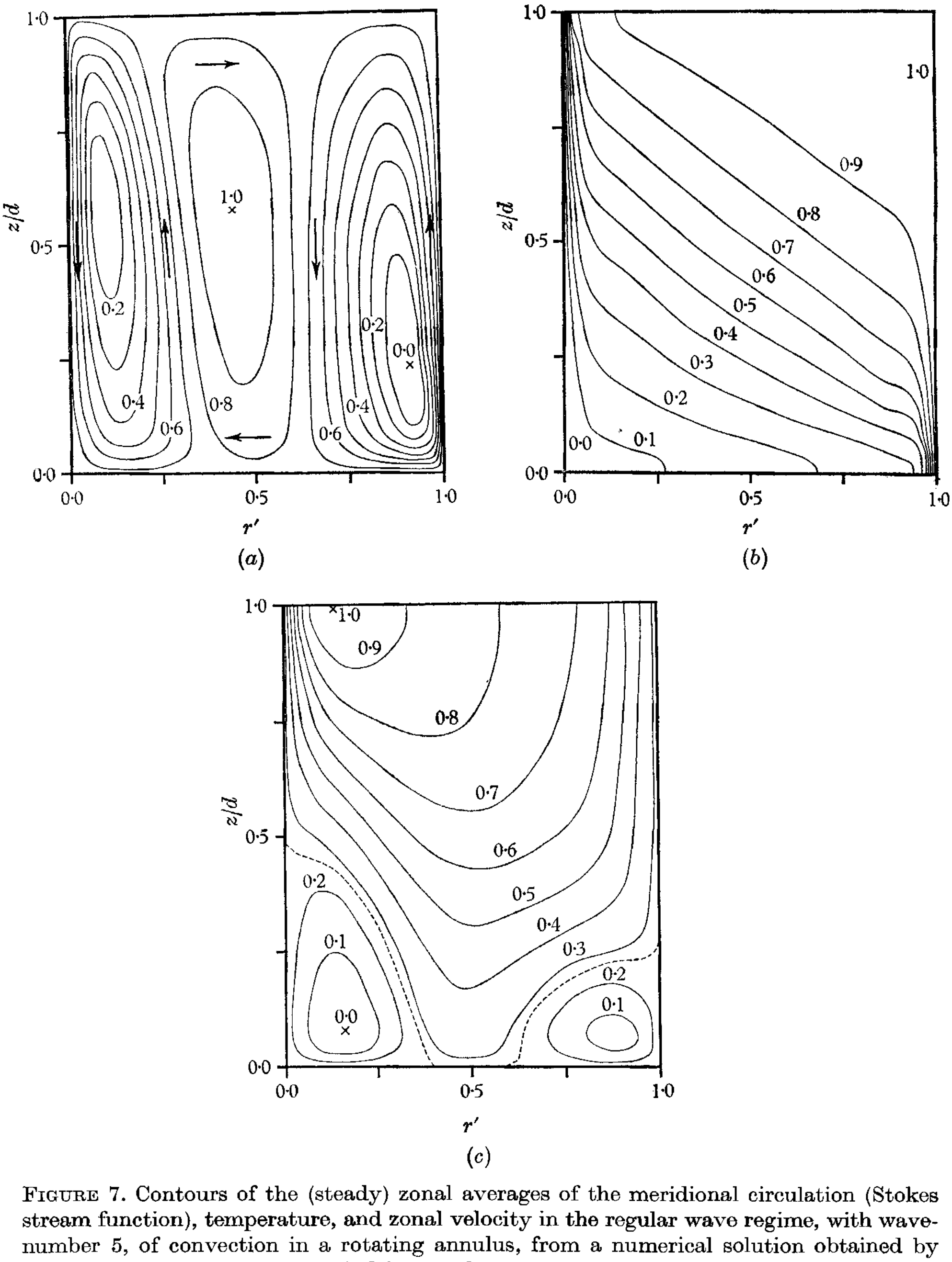


FIGURE 7. Contours of the (steady) zonal averages of the meridional circulation (Stokes stream function), temperature, and zonal velocity in the regular wave regime, with wavenumber 5, of convection in a rotating annulus, from a numerical solution obtained by Williams (1969). The inner cylindrical cold wall $r = 2$ cm ($r' = 0$) is on the left of the meridional cross-section and is held at 17·5° C; the outer hot wall $r = 5$ cm ($r' = 1$) on the right, is held at 22·5° C. The contours are evenly spaced, between $-0{\cdot}05347$ and $0{\cdot}01759$ cm^3 sec^{-1} in (*a*), and between $-0{\cdot}1090$ and $+0{\cdot}3027$ cm sec^{-1} in (*c*). The thermally insulated top and bottom boundaries are 3 cm apart; the top is stress-free while all the other boundaries are no-slip. The rate of rotation is 0·8 rad sec^{-1}, and the viscosity, thermal diffusivity and expansion coefficients are $1{\cdot}008 \times 10^{-2}$ cm^2 sec^{-1}, $1{\cdot}420 \times 10^{-3}$ cm^2 sec^{-1}, and $2{\cdot}054 \times 10^{-4}$ °C^{-1}.

The versatility of this kind of method, within its range of validity, is not confined to the present problem; it is formally capable of handling any non-singular perturbation to *any* well-posed conventional, or unconventional, differential-equation eigenvalue problem. In the present problem, for instance, one could incorporate the corrections to (2.1) representing higher approximations to Euler's equations of motion, as has recently been done for the non-geostrophic effects in an independent study by Derome & Dolph (1969). In the latter type of connexion especially, the automatic elimination of physically different modes from consideration (e.g. inertia–gravity waves) is often an advantage, rather than otherwise.

As well as the calculations of § 7, the results of § 5 on perturbed stability properties bear upon the theoretical interpretation of the baroclinic wave motions found in the heated rotating *annulus experiments* (Fowlis & Hide 1965). The lack of a well-defined inviscid short wave cutoff for a wide class of profiles $u(y,z)$ (as in (*b*), figure 3) shows that the concept of an inviscid limit for the so-called upper transition curve is probably not well-defined theoretically. (It should be mentioned that, worse still, recent theoretical considerations relevant to the symmetric régime above the transition curve have indicated that the inviscid limiting behaviour of the basic symmetric flow itself is exceedingly pathological in many cases of interest (see McIntyre 1969*b*).)

Two points are worth making about the non-uniform validity over $0 \leqslant z \leqslant 1$ of the φ representation when $k \geqslant k_N$. One is that it is possible, although cumbersome, to modify the perturbation scheme so as to give a uniformly valid representation, if φ_l and c_l are allowed to depend on μ (McIntyre 1967, p. 202). We then lose the convenience of having true power series in μ. The other is that in any case, the non-uniformity resulting from the present scheme may be quite mild in practice, and the φ representation still useful. For a recent exploitation of that, see McIntyre (1969*c*).

In § 6 we discussed the first-order effect of the horizontal shear of a profile of form $u = z+\mu u_1(y)$ upon a growing Eady wave, and in § 7 we presented numerical calculations to show that the $O(\mu)$ trend persists up to $\mu = 0{\cdot}5$ for the case $u_1 = \sin^2 \pi y\,(0 \leqslant y \leqslant 1)$, and that it does in that case give a qualitative idea of the actual mean flow change. Differential advection by any given reasonably smooth $u_1(y)$ always brings about an $O(\mu)$ eddy stress that transports zonal momentum against the mean flow gradient u_{1y}, *if* the motion is not too closely constrained laterally. (But there is at least one kind of profile, namely parabolic y-dependence, for which the latter proviso is not necessary.)

A qualitatively similar behaviour of the eddy stress is a feature of the recent results of Stone (1969) for a model whose lateral constraints are deliberately made 'slight'; his basic approximation scheme, in contrast to ours, requires u_y to be small at the outset, via an assumption that the y-scale of the basic flow $u(y,z)$ is large (as in Miles 1964). Here 'large' implies comparison with the scale NH/f of the fastest growing waves, which emerges naturally as a y-scale, as well as an x-scale, from Stone's analysis. His velocity profiles are different from ours, the horizontal shear being confined to the upper half of the two-level model used.

Finally, it should not be forgotten that by our particular choice of zero-order solution we have confined our attention to a particular kind of instability. The reader is referred to Brown (1969*a*) for some interesting results not subject to that restriction.

Much of this research was carried out while the writer was a research student at Cambridge under F. P. Bretherton, whose guidance, criticism and encouragement are gratefully acknowledged. The writer also thanks J. W. Elder for the use of certain computer programs developed by him, and the Cambridge University Mathematical Laboratory for the use of their computing facilities. Opportunities to discuss aspects of the work with J. Pedlosky and J. A. Brown, and access to unpublished results of Brown, were greatly appreciated. The work was supported by a Commonwealth Scholarship awarded by the Commonwealth Scholarship Commission in the United Kingdom, and was also supported by National Science Foundation Grant GA-402X at the M.I.T. Meteorology Department.

Appendix A. Theoretical background

(i) *Dynamics of the baroclinic instability*

The following brief description is not novel, nor is it intended to replace discussions such as those given by Bretherton (1966*b*) and, from a somewhat different point of view, by Holmboe (1959). But it quickly makes the instability plausible, and gives a substantially correct feel for the dynamics.

As was said in § 2, the fluid is stably stratified with buoyancy frequency N, but possesses available potential energy associated with a small slope $(\partial z/\partial y)_\rho$ of the lines of constant density in a meridional or yz plane. To fix ideas, suppose first that $(\partial z/\partial y)_\rho$ is positive and constant.

Imagine an initial disturbance involving a small transverse horizontal velocity v' with a wave-like x-dependence, wavelength $2\pi L_w$ say. If, hypothetically, buoyancy forces represented the only constraint on the disturbance, fluid elements drifting to the left or to the right of the main current would just tend to move along the sloping constant-density surfaces. But other effects are of course present; it is only because of one of them, the Coriolis force, that the undisturbed constant-density surfaces can slope at all. If these other effects were such as to make fluid elements move *more nearly horizontally*, i.e. on paths with positive slope less than $(\partial z/\partial y)_\rho$, then potential energy could clearly be released. The buoyancy force could do work against whatever was causing the fluid particles to move on their shallower trajectories.

Now the instability is possible because sideways-drifting fluid elements can indeed be made to move along such paths, in the simple situation we are considering, by a combination of two things. The first is the presence of rigid boundaries that are either horizontal, or nearly so, with slope less than $(\partial z/\partial y)_\rho$. The second is the resistance to horizontal divergence that arises from a sufficiently strong Coriolis effect f; this 'rotational stiffness' has the effect of making the kinematical constraint due to a boundary felt throughout a substantial depth ($\propto f$) of fluid (although not in the simple Taylor–Proudman sense appropriate to a homogeneous fluid). The penetration height scale is in fact fL_w/N (Walin 1969).

Note that if $(\partial z/\partial y)_\rho$ were allowed to vary with height, a level where $(\partial z/\partial y)_\rho$ is relatively small could play the same role as a rigid boundary (see Green 1960, §8; McIntyre 1969*c*).

In either case, it is the resulting kinematical-rotational constraint that gives rise to the pressure field against which the buoyancy force is enabled to do work. From the point of view of vorticity, the buoyancy force can be thought of as slowly stretching or compressing the very strong tubes of absolute vertical vorticity of the rotation f, the effect of which is described by the dominant term $f\,\partial w/\partial z$ in the vertical vorticity equation.

The work done by buoyancy, then, appears as kinetic energy of the *horizontal* relative velocities associated with the resulting 'spin-up' relative vorticity. To a first approximation, the Coriolis force does the actual accelerating. (One should note the complete contrast with e.g. Solberg's symmetric baroclinic instability (see McIntyre 1969*a* and references), a type of essentially non-geostrophic sloping convection in which buoyancy *can* contribute directly to the acceleration of a fluid element.)

The horizontal velocities produced by vortex-tube stretching can indeed reinforce the original disturbance, giving exponential growth, provided that there is an appropriate phase change with height. It is found that the surfaces of constant phase of v' must slope forwards–downwards, so that $\partial z/\partial x < 0$, i.e. they slope in the sense 'opposite to that of the velocity profile'. To see this, and to understand among other things the necessary role of differential advection by the vertical shear u_z, a more detailed description is needed (see e.g. Bretherton 1966*b*).

(ii) *The basic formulation*

This is well known (Phillips 1963; Pedlosky 1964*a*) and will be sketched only briefly, for the Boussinesq liquid case. First, the hydrostatic and geostrophic approximations are made. The latter signifies an approximate balance between horizontal pressure and Coriolis forces, the condition for which is formally expressed by the smallness of the Rossby number,

$$Ro = U/fL \ll 1, \tag{A 1}$$

where U is a characteristic horizontal velocity and L a horizontal length (taken for convenience as the channel width, in the present problem). The time scale is assumed $\gtrsim L/U$. In this approximation the departure ψ from the horizontally-averaged hydrostatic pressure becomes a stream function for the dimensionless horizontal velocities, after ψ is made dimensionless by the scale $f\rho_0 UL$, where ρ_0 is an average density for the whole (Boussinesq) fluid. The approximate velocities (scale U) are then

$$u = -\psi_y, \quad v = \psi_x, \tag{A 2}$$

in the x and y directions respectively. The vertical velocities are small of order

$(RoH/L)\,U$, but important because of vortex-tube stretching. We assume $\epsilon \equiv f^2L^2/N^2H^2 \sim 1$, in the formal limit $Ro \to 0$ implied by (A 1).

Under all the above assumptions it can be shown that N^2, and thus ϵ, can be taken as a horizontal and time average and thus as a function of height z only, and that the vorticity equation for inviscid adiabatic motion can be reduced to a single approximate equation involving the vertical component only of the dimensionless absolute vorticity, $Ro^{-1}+\psi_{xx}+\psi_{yy}+O(Ro)$:

$$\left(\frac{\partial}{\partial t}-\psi_y\frac{\partial}{\partial x}+\psi_x\frac{\partial}{\partial y}\right)[Ro^{-1}+\psi_{xx}+\psi_{yy}+(\epsilon\psi_z)_z] = 0. \tag{A 3}$$

The ϵ term represents the stretching of vertical vortex tubes. The vertical velocity is related to ψ through the adiabatic equation. To sufficient accuracy,

$$w = -\epsilon\left(\frac{\partial}{\partial t}-\psi_y\frac{\partial}{\partial x}+\psi_x\frac{\partial}{\partial y}\right)\psi_z. \tag{A 4}$$

The scale for w is $(RoH/L)\,U$. In virtue of the hydrostatic relation, $-\psi_z$ represents the local density anomaly due to the motion. In (A 3) the quantity in square brackets is related to Ertel's potential vorticity in the manner explained by Charney & Stern (1962, p. 163) and will be called a 'quasi-potential-vorticity'.

The eigenvalue problem for normal-mode disturbances to the mean flow $u(y,z)$ is now obtained by posing

$$\psi = -\int^y u(\eta,z)\,d\eta+\psi'; \quad \psi' = \mathrm{Re}\,\{\varphi(y,z)\,e^{ik(x-ct)}\} \tag{A 5}$$

where, formally, $|\psi'| \ll |\psi|$. The dimensionless wave-number k is considered real, but c and $\varphi(y,z)$ may be complex. Then (A 3) yields the linearized equation (2.1*a*). The coefficient q_y that appears in (2.1*a*) is the transverse gradient of mean quasi-potential-vorticity,

$$q_y = \beta-(\epsilon u_z)_z-u_{yy}; \tag{A 6}$$

$\beta = d(Ro^{-1})/dy \simeq$ const. is included to represent the earth's planetary vorticity gradient or north–south variation of f, where relevant. The boundary conditions for (2.1*a*) are that w and $v = \psi'_x$ vanish on horizontal and vertical boundaries respectively, which yields (2.1*b*) and (2.1*c*).

(iii) *Eady's solution*

When $u = z$, $\beta = 0$, $\epsilon(z) =$ constant, so that $q_y = 0$, it can easily be shown that (2.1) has the following closed form solutions, which were first described by Eady (1949):

$$\left.\begin{aligned} \varphi_{0m}(y,z) &= \sin m\pi y\,.\,\chi_m(z), \\ c_{0m} &= \tfrac{1}{2}\pm\tfrac{1}{2}\alpha_m^{-1}[(\alpha_m-\coth\alpha_m)(\alpha_m-\tanh\alpha_m)]^{\frac{1}{2}}, \\ \text{where}\quad \chi_m(z) &= \kappa_m c_{0m}\cosh\kappa_m z-\sinh\kappa_m z \\ \text{and}\quad \kappa_m &= 2\alpha_m = \epsilon^{-\frac{1}{2}}(k^2+m^2\pi^2)^{\frac{1}{2}} \quad (m = 1,2,\ldots). \end{aligned}\right\} \tag{A 7}$$

For each integer m these solutions represent either an amplifying-decaying pair of waves with $c_r = \mathrm{Re}\,(c_{0m}) = \frac{1}{2}$, or a pair of neutral waves ($c = c_r$, $\geqslant$ and $\leqslant \frac{1}{2}$), according as

$$\alpha_m < \quad \text{or} \quad \geqslant \alpha_N, \quad = 1{\cdot}1997 (\alpha_N = \coth \alpha_N). \tag{A 8}$$

Thus there is a short-wave cut-off to the instability, at a critical neutral wave-number $k = k_N = (4\epsilon\alpha_N^2 - m^2\pi^2)^{\frac{1}{2}}$, if k_N is real. For any m such that k_N is real, there is a well-marked maximum growth rate $kc_i\,[c_i \equiv \mathrm{Im}\,(c_{0m})]$ at some $k < k_N$: $k = k_M$, say. The largest of these maxima occurs, if any occur, for $m = 1$. Some numerical values for $m = 1$ are shown in table 1.

ϵ/π^2	ϵ	k_M	κ	c_i	$k_M c_i$
0·9119	9	4·392	1·800	0·1677	0·7366
1·6211	16	6·117	1·719	0·1787	1·0931

TABLE 1

For the long waves in the atmosphere (wavelength 6000 km), these dimensionless maximum growth rates typically correspond to doubling times in the vicinity of 2 days. The structure in an xz plane of an unstable wave is shown in figure 8, from Eady (1949).

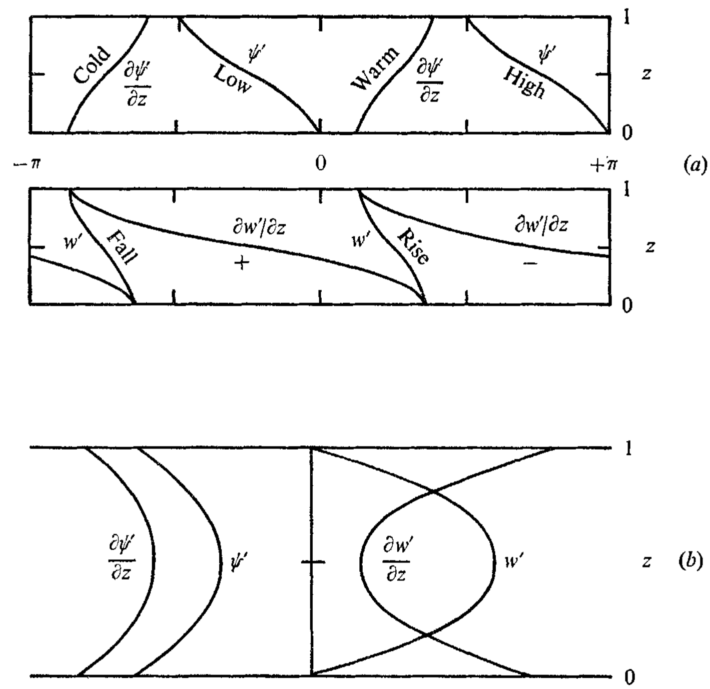


FIGURE 8. (a) Negative relative phases and (b) amplitudes of an amplifying baroclinic wave on the simple velocity profile $u = z$, from Eady (1949). The negative-phase diagrams give the actual side view of the wave if the x axis points toward the right.

Note that an unstable $\chi_m(z)$ possesses symmetry about $z = \frac{1}{2}$ that is obscured by the otherwise convenient form given. If we define a symmetrical co-ordinate $z_s = z - \frac{1}{2}$, then χ_m is equal to a complex constant times

$$\cosh \kappa_m z_s + (\text{imaginary constant}) \times \sinh \kappa_m z_s. \tag{A 9}$$

At a critical neutral point $k = k_N$ we have $c_{0m} = \frac{1}{2}$, and

$$\varphi_{0m} = \varphi_{0m}^N \equiv \sin m\pi y \,.\, (\alpha_N^2 - 1)^{\frac{1}{2}} \cosh 2\alpha_N z_s. \tag{A 10}$$

The short neutral waves ($k > k_N$) are asymmetrical. For $k \gg k_N$, each is associated with one horizontal boundary exclusively, because of a small penetration height scale fL_w/N, $\ll H$.

Note that if generalized functions are admitted as solutions, there is also, for each k, a continuous spectrum of singular neutral modes with $0 \leqslant c \leqslant 1$ (Pedlosky 1964c). At $z = c$, φ_z has a jump discontinuity. These resemble the singular modes discovered by Rayleigh (1895) and used by Orr (1907) to solve the initial value problem for small disturbances to plane Couette flow.

The following identities are useful; we drop the suffix m:

$$\kappa^2 c_0(1-c_0) + 1 = \kappa \coth \kappa, \tag{A 11}$$

$$\{1 - \kappa^2 c_0^2\}\{1 - \kappa^2(1-c_0)^2\} = \kappa^2 \operatorname{cosech}^2 \kappa, \tag{A 12}$$

$$\int_0^1 dy \left[\frac{\{\varphi_0(y,z)\}^2}{(z-c_0)^2}\right]_{z=0}^{z=1} = \frac{\kappa^4(c_0 - \frac{1}{2})}{\kappa^2(1-c_0)^2 - 1}, \tag{A 13}$$

$$\int_0^1\int_0^1 \{\varphi_0^N(y,z)\}^2\, dy\, dz = \tfrac{1}{4}\alpha_N^2, \tag{A 14}$$

$$\frac{\chi(1)\,\tilde{\chi}(1-z)}{\kappa(1-c_0)} = \chi(z), \tag{A 15}$$

where $\tilde{\chi}$ denotes the result of replacing c_0 in χ by $(1-c_0)$.

Appendix B

(i) *Definition of* B_l, I_l

It is convenient first to substitute the expansion (3.2b) into (2.1a,b), divide by $(z-c_0)$, and rearrange. (If c_0 is real it is necessary to assume that u_1 is analytic in z and to go into the complex plane, as discussed in § 4.) There results

$$\left.\begin{aligned} L(\varphi) &\equiv \varphi_{zz} + \epsilon^{-1}(\varphi_{yy} - k^2\varphi) \\ &= \frac{-\mu u_1 + \mu c_1 + \mu^2 c_2 + \dots}{z - c_0} L(\varphi) - \frac{\mu\epsilon^{-1} q_{1y}}{z-c_0}\varphi \\ \text{and on } z = 0, 1, \quad D(\varphi) &\equiv \varphi_z - \frac{\varphi}{z-c_0} \\ &= \frac{-\mu u_1 + \mu c_1 + \mu^2 c_2 + \dots}{z-c_0}\varphi_z + \frac{\mu u_{1z}}{z-c_0}\varphi. \end{aligned}\right\} \tag{B 1}$$

We then introduce (3.2a) and require that (B 1) be satisfied separately at each order μ^l ($l = 0, 1, \ldots$), giving (3.3). The right-hand sides I_l, B_l are defined recursively by $I_0 = B_0 = 0$ and, for $l \geqslant 1$,

$$(z-c_0)\,I_l = \sum_{j=1}^{l-1} c_j I_{l-j} - u_1 I_{l-1} - \epsilon^{-1} q_{1y}\varphi_{l-1}, \tag{B 2a}$$

$$(z-c_0)\,B_l = (z-c_0)\,B_l' + (z-c_0)^{-1} c_l \varphi_0, \tag{B 2b}$$

where
$$(z-c_0)\,B_l' = \sum_{j=1}^{l-1} c_j \left\{B_{l-j} + \frac{\varphi_{l-j}}{z-c_0}\right\} - u_1\left\{B_{l-1} + \frac{\varphi_{l-1}}{z-c_0}\right\} + u_{1z}\varphi_{l-1}. \tag{B 2c}$$

$\left(\sum_{j=1}^{0} \text{gives zero, by convention.}\right)$ For convenience, derivatives of φ_l have been eliminated at each stage, by the use of (3.3), as happens to be possible in this problem.

(ii) *The generalized Green's function*

It is a straightforward task to obtain the solution to (3.5) and (3.6) as a sine series in y, whose coefficients are functions of z with discontinuities in their first derivatives at $z = \zeta$. Note that $\mathfrak{G}$ depends on m, both through the operator D, in which the value of c_0 is given by (A 7), and through the presence of φ_0 in (3.5a) and (3.6); thus we write

$$\mathfrak{G} = \mathfrak{G}_m(y,z;\eta,\zeta) = 2\sum_{n=1}^{\infty} \sin n\pi y \sin n\pi\eta\, G_m^n(z;\zeta). \tag{B 3a}$$

For $n \neq m$, it is found that

$$G_m^n(z;\zeta) = \frac{[\kappa_n c_{0m}\cosh\kappa_n z_< - \sinh\kappa_n z_<]\,[\kappa_n(1-c_{0m})\cosh\kappa_n(1-z_>) - \sinh\kappa_n(1-z_>)]}{\kappa_n^3 \sinh\kappa_n[c_{0m}(1-c_{0m}) + \kappa_n^{-2} - \kappa_n^{-1}\coth\kappa_n]}, \tag{B 3b}$$

where
$$z_< = \min(z,\zeta), \quad z_> = \max(z,\zeta), \tag{B 3c}$$

with the obvious interpretation if z and ζ both lie on a complex contour Γ_z such as that in figure 1, and where (cf. (A 7))

$$\epsilon\kappa_n^2 = k^2 + n^2\pi^2.$$

This is a one-dimensional Green's function in the usual sense. For $n = m$,

$$\begin{aligned} G_m^m(z;\zeta) = {} & A'\chi_m(z)\,[\kappa_m c_{0m}^*\zeta\sinh\kappa_m\zeta - (\zeta - c_{0m})\cosh\kappa_m\zeta] \\ & - (1/\kappa_m)\,\chi_m(z_<)\cosh\kappa_m z_> + \chi_m(\zeta)\times\text{func}\,(z), \end{aligned} \tag{B 3d}$$

where
$$\begin{aligned} A' &= \left\{2\kappa_m \int_0^1 \chi_m\chi_m^*\,dz\right\}^{-1}, \\ &= \begin{cases} \dfrac{1}{\kappa_m^3|c_{0m}|^2} = \dfrac{1}{\kappa_m^3 c_{0m}(1-c_{0m})}, & \text{if } c_{0m} \text{ is complex,} \\[2ex] \dfrac{\kappa_m^2(1-c_{0m})^2 - 1}{\kappa_m^3[\kappa_m^2 c_{0m}^2(1-c_{0m})^2 + 3c_{0m}(1-c_{0m}) - 1]}, & \text{if } c_{0m} \text{ is real.} \end{cases} \end{aligned}$$

The contribution to G_m^m not explicitly written out may be ignored when $\mathfrak{G}_m$ is used as in (3.4b), that is, with ζ as the dummy variable, in virtue of the solubility condition. (The extra contribution consists of the symmetrizing complement of the first line of (B 3d), plus a further term proportional to $\chi_m(z)\,\chi_m(\zeta)$.)

Appendix C. The expansion at the Eady neutral point $k = k_N$

Posing the expansions (3.7) leads to the sequence, dropping the superscript N,

$$\begin{cases} L(\varphi_l) = I_l, & \text{(C 1}a\text{)} \\ D(\varphi_l) = B_l \quad \text{on} \quad z = 0, 1, & \text{(C 1}b\text{)} \\ \varphi_l = 0 \quad \text{on} \quad y = 0, 1, & \text{(C 1}c\text{)} \end{cases}$$

in which, this time, $I_0 = I_1 = B_0 = 0$, and $z_s B_1 = c_1 \varphi_{0z}$, where $z_s \equiv z - \frac{1}{2}$. The formula (A 10) gives φ_0. The first-order problem is now automatically soluble for φ_1, because of the same symmetry that was associated with the breakdown of the μ expansion. In fact, noting that (C 1a) for $l = 1$ is satisfied by φ_{0z}, and recalling (A 8), we find that

$$\varphi_1 = \frac{c_1 \varphi_{0z}}{\alpha_N^2 - 1}, \quad = \frac{2\alpha_N c_1}{(\alpha_N^2 - 1)^{\frac{1}{2}}} \sin m\pi y \,.\, \sinh 2\alpha_N z_s. \tag{C 2}$$

A point of interest that now emerges is that c_1 will not be determined until the problem for φ_2 is considered, and so on.

In the same way as before, recursion formulae give I_l and B_l for $l \geqslant 1$. With the convention $\varphi_{-1} = I_{-1} = I_0 = B_0 = 0$, we can show that for $l \geqslant 1$

$$z_s I_l = \sum_{j=1}^{l-2} c_j I_{l-j} - u_1 I_{l-2} - \epsilon^{-1} q_{1y} \varphi_{l-2}, \tag{C 3a}$$

$$z_s B_l = z_s B_l^{(d)} + c_l \varphi_{0z}. \tag{C 3b}$$

(As before, summations with reversed limits are zero; (C 1a) has been used.) $B_l^{(d)}$ will be defined below, by (C 3c, d). It is zero for $l = 1$, and for $l \geqslant 2$ it will involve c_{l-1} but not c_l, and will turn out to be completely determined at the current (lth) stage. *The term $c_l \varphi_{0z}$ will not be determined, since c_l disappears from* the solubility condition by symmetry, as before. That condition is

$$-\iint \varphi_0 I_l \, dy\, dz + \int dy \left[\varphi_0 B_l^{(d)}\right]_{z=0}^{z=1} = 0. \tag{C 4}$$

But we are free to choose c_{l-1}, since by the same token it could not have been determined previously. It will turn out that (C 4) can always be satisfied by just one such choice (apart from a sign ambiguity when $l = 2$). The solution φ_l which then exists can be split into two parts. One arises from the c_l term in (C 3b), and is just c_l/c_1 times (C 2). The remaining part $\varphi_l^{(d)}$ is independent of c_l, and will thus be determinate at the present stage:

$$\varphi_l^{(d)} = -\iint \mathfrak{G} I_l(\eta, \zeta)\, d\eta\, d\zeta + \int d\eta \left[\mathfrak{G} B_l^{(d)}(\eta, \zeta)\right]_{\zeta=0}^{\zeta=1}. \tag{C 5}$$

The whole solution, then, is

$$\varphi_l = \varphi_l^{(d)} + \frac{c_l \varphi_{0z}}{\alpha_N^2 - 1}. \tag{C 6a}$$

This agrees with (C 2), since $\varphi_1^{(d)} = 0$.

We can now complete the recursive definition of B_l by supplying the definition of $B_l^{(d)}$. It will be convenient to define first, for $l \geqslant 2$,

$$z_s B_l^{(c)} = c_1(B_{l-1}^{(d)} + z_s^{-1}\varphi_{l-1}^{(d)}) + \sum_{j=2}^{l-2} c_j(B_{l-j} + z_s^{-1}\varphi_{l-j}) - u_1(B_{l-2} + z_s^{-1}\varphi_{l-2}) + u_{1z}\varphi_{l-2}. \tag{C 3c}$$

Then, using (C 6a) above, (C 3b), (C 2), and (C 1b), we may define $B_l^{(d)}$ and thence B_l by

$$\left.\begin{aligned} B_1^{(d)} &= 0, \\ z_s B_l^{(d)} &= z_s B_l^{(c)} + h_l c_1 c_{l-1}\left(\frac{\alpha_N^2}{\alpha_N^2 - 1}\right)\frac{\varphi_{0z}}{z_s} \quad (l \geqslant 2), \end{aligned}\right\} \tag{C 3d}$$

where

$$h_l = \begin{cases} 1 & \text{if} \quad l = 2, \\ 2 & \text{if} \quad l \geqslant 3. \end{cases}$$

Finally (C 4) can be re-written, after a little more manipulation, to give c_{l-1} explicitly:

$$c_1 c_{l-1} = \frac{\alpha_N^2 - 1}{8h_l \alpha_N^4}\left\{\iint \varphi_0 I_l \, dy\, dz - \int dy \left[\varphi_0 B_l^{(c)}\right]_0^1\right\} \quad (l \geqslant 2). \tag{C 6b}$$

The expansions (3.7) are now completely defined by (C 6a, b), the recursive definitions (C 3a–d), and the definition (A 10) of φ_0. It may be verified that the formulae are all explicit at each stage. (Note that $B_2^{(c)}$ does not depend on c_1.) After the lth stage we know $\varphi_0, \varphi_1, \ldots, \varphi_{l-1}, \varphi_l^{(d)}$, and $c_0, c_1, \ldots, c_{l-1}$ (but not c_l).

Appendix D. Some mathematical details

We give here some of the details, for $k \neq k_N$, of the mathematical justification that is possible using elementary analysis. First, uniform convergence† is established for $|\mu| <$ some sufficiently small $\mu_0 > 0$. Then we prove that the result of substituting the series back into (2.1) is meaningful, and thence that the series do actually represent a solution of (2.1) for appropriate u and q_y of the form (3.1).

We thereby prove, incidentally, the existence of solutions to (2.1), under much more general conditions than those permitting explicit separable solutions.

When c_0 is not real ($k < k_N$ in the present problem), nothing is assumed about $u_1(y, z)$ except sufficient differentiability, implying boundedness in the closed domain of the problem. When c_0 is real ($k \geqslant k_N$), u_1 is further assumed analytic in z, and the domain of the eigenvalue problem and of the various integrals understood to be some suitable complex $\mathscr{D}$ (§ 4).

† The author did not see the possibility of a straightforward proof of convergence until after the completion of much of the work reported in this paper, when he came across the essential idea in the book by Titchmarsh (1958, p. 226). Titchmarsh also states a perturbation formula that amounts to a generalized Green's representation ((19.5.5), p. 224), although he does not indicate either its conceptually simple nature, or its practical importance for non-standard types of eigenvalue problem.

(i) *Uniform convergence in $\mathscr{D}$ for sufficiently small* $|\mu|$

The proof is straightforward, although to obtain sharp estimates for the radius of convergence of, say, the c expansion, and to avoid obscuring the fact that the latter cannot, obviously, depend upon details of the choice of $\mathscr{D}$ if u_1 is analytic, one would have to use methods deeper than the direct one used here. Therefore no attempt is made to estimate particular radii of convergence numerically, or to construct refined inequalities.

The essence of the proof is to start by considering a series like $\Sigma a_l \mu^l$, where the a_l are defined recursively by specifying $a_1 > 0$ and then defining

$$a_l = M \sum_{j=1}^{l-1} a_j a_{l-j} + N a_{l-1} \quad (l > 1), \tag{D 1}$$

M and N being positive constants. The radius of convergence of $\Sigma \alpha_l \mu^l$ is at least $1/(4Ma_1 + 2N)$ as will now be shown.

First, consider the function

$$g(\mu) = \frac{1}{2M}\left[1 - \left\{1 - \frac{\mu}{\rho}\right\}^{\frac{1}{2}}\right], \tag{D 2}$$

which has the expansion

$$g = \sum_{l=1}^{\infty} d_l \mu^l,$$

say, whose radius of convergence is evidently ρ. Now from (D 2),

$$g^2 = -\frac{\mu}{4M^2\rho} + \frac{g}{M},$$

$$= -\frac{\mu}{4M^2\rho} + \frac{1}{M} \sum_{l=1}^{\infty} d_l \mu^l \quad (\mu < \rho). \tag{D 3}$$

But for $\mu < \rho$ the coefficients of the above power series expansion of g^2 may also be obtained by multiplying the (absolutely convergent) series for g by itself. Comparison with (D 3) then gives the relations $d_1 = (4M\rho)^{-1}$ and

$$d_l = M \sum_{j=1}^{l-1} d_j d_{l-j} \quad (l \geqslant 2). \tag{D 4}$$

This shows that the power series whose coefficients are *defined* by (D 4), with d_1 specified, has radius of convergence $\rho = (4Md_1)^{-1}$. (The function g was of course arrived at in the first place by considering the formal product $(\Sigma d_l \mu^l)^2$.)

If now we choose $d_1 = (a_1 + N/2M)\,(> 0)$, then (D 1) and (D 4) imply

$$(a_1 \leqslant d_1), \quad a_2 \leqslant d_2; \quad a_3 \leqslant d_3, \ldots.$$

Therefore the radius of convergence of $\Sigma a_l \mu^l$, where the positive coefficients $a_2, a_3, \ldots$ are related to a_1 by (D 1), is at least

$$\frac{1}{4Ma_1 + 2N},$$

as was asserted. (Clearly this is already a crude estimate.) To apply these ideas to prove convergence of $\Sigma\mu^l\varphi_l$, $\Sigma\mu^l c_l$, we construct first the series $\Sigma a_l\mu^l$, $\Sigma b_l\mu^l$, defined by a scheme slightly more general than (D 1), of the form

$$\left.\begin{aligned} a_l &= M_a^{(1)}\Sigma a_j a_{l-j} + M_a^{(2)}\Sigma a_j b_{l-j} + M_a^{(3)}\Sigma b_j b_{l-j} + N_a^{(1)} a_{l-1} + N_a^{(2)} b_{l-1}, \\ b_l &= M_b^{(1)}\Sigma a_j a_{l-j} + M_b^{(2)}\Sigma a_j b_{l-j} + M_b^{(3)}\Sigma b_j b_{l-j} + N_b^{(1)} a_{l-1} + N_b^{(2)} b_{l-1}, \end{aligned}\right\} \tag{D 5}$$

where $a_1 > 0$, $b_1 > 0$ have been specified, and the summations are taken from $j = 1$ to $j = l-1$. It is then easy to verify that the radii of convergence of the series $\Sigma a_l\mu^l$, $\Sigma b_l\mu^l$ are each at least

$$\left.\begin{aligned} &\frac{1}{4M\max(a_1, b_1) + 2N}, \\ \text{where} \qquad M &= \sum_{i=1}^{3} \max(M_a^{(i)}, M_b^{(i)}), \\ N &= \sum_{i=1}^{2} \max(N_a^{(i)}, N_b^{(i)}). \end{aligned}\right\} \tag{D 6}$$

We are now ready to consider the formulae (3.4), (B 2), giving φ_l and c_l. A hat over a symbol, as in $\hat{I}_l$, will indicate an upper bound, taken over all (y, z) in $\mathscr{D}$ where relevant: for example, $|I_l| \leqq \hat{I}_l$ (= const.). In estimating the Green's representations (3.4*b*) we note that although $\mathfrak{G}(y, z; \eta, \zeta)$ has a logarithmic infinity at $(\eta, \zeta) = (y, z)$, its integral with respect to η or y is finite, and likewise the integral of $|\mathfrak{G}|$. Indeed, there are finite *constants* G_I, G_B such that for all (y, z) in $\mathscr{D}$

$$\iint |\mathfrak{G}|\, d\eta\, d\zeta \leqslant G_I, \quad \int d\eta \sum_{\zeta=0}^{1} |\mathfrak{G}| \leqslant G_B. \tag{D 7}$$

Then from (3.4*b*)

$$|\varphi_l| \leqslant G_I \hat{I}_l + G_B \hat{B}_l \equiv \hat{\varphi}_l, \quad \text{say, appropriately.} \tag{D 8}$$

The other bounds $\hat{c}_l$, $\hat{I}_l$, $\hat{B}_l$, $\hat{B}'_l$, are defined in a similar way, using the straightforward estimates that can be written down from (B 2) and (3.4). Writing $Z \equiv (z - c_0)^{-1}$ and using (D 8) to eliminate reference to $\hat{\varphi}_l$, we have the following relations, which define the bounds recursively:

$$|c_l| \leqslant \hat{c}_l \equiv E\hat{\varphi}_0(\hat{I}_l + 2\hat{B}'_l), \tag{D 9}$$

$$|I_l| \leqslant \hat{I}_l \equiv \hat{Z}\left[\sum_{j=1}^{l-1} \hat{c}_j \hat{I}_{l-j} + (\hat{u}_1 + \epsilon^{-1}\hat{q}_{1y} G_I)\hat{I}_{l-1} + \epsilon^{-1}\hat{q}_{1y} G_B \hat{B}_{l-1}\right], \tag{D 10}$$

$$\begin{aligned} |B'_l| \leqslant \hat{B}'_l \equiv{}& \hat{Z}^2 G_I \sum_{j=1}^{l-1} \hat{c}_j \hat{I}_{l-j} + \hat{Z}(\hat{Z}G_B + 1)\sum_{j=1}^{l-1} \hat{c}_j \hat{B}_{l-j} \\ &+ \hat{Z}G_I(\hat{Z}\hat{u}_1 + \hat{u}_{1z})\hat{I}_{l-1} + \hat{Z}\{\hat{u}_1 + G_B(\hat{Z}\hat{u}_1 + \hat{u}_{1z})\}\hat{B}_{l-1}, \end{aligned} \tag{D 11}$$

$$|B_l| \leqslant \hat{B}_l \equiv \hat{Z}^2\hat{\varphi}_0^2 E\hat{I}_l + (1 + 2\hat{Z}^2\hat{\varphi}_0^2 E)\hat{B}'_l, \tag{D 12}$$

where
$$E \equiv \left|\frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})}\right|.$$

In (D 12), reference to $\hat{c}_l$ has been eliminated by means of (D 9).

All we need do now is to note that the pair of recursion relations (D 10) and (D 11) are of the form (D 5), after elimination of $\hat{c}_j$ and $\hat{B}_j$ using (D 9) and (D 12).

(If $\hat{I}_l$ is identified with a_l and $\hat{B}'_l$ with b_l, then $M^{(3)}_{\alpha}$ is zero but the remaining M's and N's are not.) This shows that $\hat{I}_l$ and $\hat{B}'_l$ and, in virtue of (D 8), (D 9), and (D 12), $\hat{\varphi}_l$ and $\hat{c}_l$, are the (constant) coefficients of majorant series with finite (and constant) radii of convergence. Thus uniform convergence is proved, for $|\mu| <$ some finite μ_0.

(ii) *Proof that the series solve the perturbed problem*

The analysis just given can be extended to prove that $\{\Sigma\mu^l\varphi_l, \Sigma\mu^l c_l\}$, for any $|\mu| <$ some finite μ_0, does solve the eigenvalue problem associated with $u = z+\mu u_1$, $q_y = \mu q_{1y}$. (Note the corollary that existence of a solution is then proved.)

The formalism already ensures that the series satisfy the equation and the boundary conditions term by term. It is sufficient, then, to show that all the infinite series that arise on back substitution are absolutely and uniformly convergent over $\mathscr{D}$, since they are then immediately meaningful in the context of the boundary-value problem.† For the z boundary condition this follows from the term-by-term balance, since there is only one series involved, $\Sigma\mu^l\varphi_{lz}$, whose convergence has not been investigated. But in connexion with the differential equation we must show independently that one of $\Sigma|\mu^l\varphi_{lyy}|$, $\Sigma|\mu^l\varphi_{lzz}|$ is uniformly convergent. It would be straightforward, if tedious, to do this by extending the foregoing proof to include bounds on $|\varphi_{ly}|$ and $|\varphi_{lyy}|$ say, as well as on $|\varphi_l|$. A little more care is needed in estimating the Green's representations for the derivatives; bounds on the first and second y-derivatives of u_1 and q_{1y} will now be involved.

Alternatively, suppose that $u_1(y,z)$ is an analytic function of z whose singularities are bounded away from Γ_z, uniformly over $\mathscr{D}$. Then, since also (y, c_0) can be supposed bounded away from $\mathscr{D}$, the formulae show by induction that $\varphi_l(y,z)$ is analytic in z and that its z-singularities are also bounded away from Γ_z, uniformly in $\mathscr{D}$, and in l also. Thus for any z' on Γ_z, φ_l has an expansion in powers of $(z-z')$ whose radius of convergence $\geqslant$ some number that is non-zero and independent of l as well as of y and z'. Also, by a trivial extension of the previous analysis, $\Sigma\mu^l\varphi_l(z)$ has μ-radius of convergence greater than some constant μ_0, for any z within some neighbourhood of z'. Therefore $\Sigma\mu^l\varphi_l$ can be further expanded as a double series in powers of μ and $(z-z')$, absolutely convergent for $|\mu| < \mu_0$ and small but finite $|z-z'|$. Term-by-term differentiation with respect to z, holding μ at any value within $|\mu| < \mu_0$, then gives the second (or any) derivative with respect to z near $z = z_1$, as another absolutely convergent double power series. Since this can be rewritten as $\Sigma\mu^l\varphi_{lzz}$, the latter must also be absolutely convergent for $|\mu| < \mu_0$, when $z = z'$ in particular. The convergence is uniform over $\mathscr{D}$ as required, since μ_0 can be taken independent of y and z'.

† If the perturbation method were being used to account for higher approximations to the equations of motion, justification would not be quite so straightforward. In the independent analysis of non-geostrophic effects by Derome & Dolph (1969), for instance, the boundary conditions force non-uniformity of convergence at corners such as $y = z = 0$. Although the series are not then immediately meaningful globally, one would still expect pointwise convergence to a solution of the full problem. Indeed, under suitable assumptions, this would follow from considerations of analytic continuation in μ.

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